The method of colouring in combinatorics

Vilgot Jansson, Viggo Laaksoharju, David Mörtberg,
Pontus Nilsson, Ferdinand Rönngren (Baldergymnasiet)
Rosendalsgymnasiet, Uppsala

May 2020

Contents

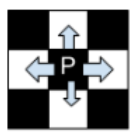
1 Colouring proof

In this section we will present the colouring proof technique. We will go through how and where colouring proofs are effective and what they can and can not do. The second section is dedicated to presenting a process that can be used to find useful patterns when presented with a colouring problem and lastly, in section 3, we present some less known methods that are closely related to colouring proofs and can be used in similar problems. Accompanying these sections will be several examples to show how they are used.

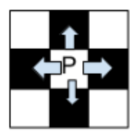
1.1 What is a colouring proof?

A colouring proof is a sort of invariant proof which can mainly be used to prove that something isn't possible. The essence of invariant proofs is to strip the problem of any unnecessary details and only keep the information that best describes why something isn't possible, making it very easy to follow invariant proofs. They do this by finding something that is invariable, constant, doesn't change over the course of the problem, making everything that would require this constant to change, be impossible. colouring proofs are invariant proofs that have invariants that show up when you colour in the correct pattern. Making them a perfect method to tackle tiling problems, which can easily hold patterns. Here, next we have a simple example that demonstrates the power of colouring. A commonly used type of colouring is chessboard colouring. An easy example:

Problem 1. Is it possible to make a walk on an 8×8 board with squares such that you walk on each square exactly once such that your start and end positions are in opposite corners? You are only allowed to walk to an adjacent square.



The player standing on a black square



The player standing on a white square

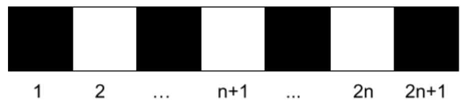
Solution. By colouring the board in a chessboard pattern we notice that while standing on a white square we may only move to a black one and vice versa. The starting corner can be either black or white, it does not matter. There are 63 squares left to walk on, which means 63 moves remain. If we start in a white corner, our first move will be to a black square. Our second move is to a white square and our third is to a black square. We can see that every odd move is to a black square and every even move is to a white square. This fact is our invariant. No matter how we move, every odd move will still be to a black square and every even move will still be to a white square. We make 63 moves, so our last move is to a black square. However, opposite corners have the same colour, namely white, so we cannot end in the opposite corner of the one we started in. □

The power of colouring is shown here by discarding an immense amount of detail from the problem. Normally you would think that the player has a lot of freedom by choosing to go up, down, right or left. But by a colour argument we see that the player does not have any freedom whatsoever (at least not as much as we initially anticipated), since it has only one single option. Either to move from a black square to a white square, or if it is on a white square, move to a black square. By simplifying the problem to the core reason why it is impossible, the proof will be much easier to write and read.

A type of alternating pattern can be used for placing different pieces on a line. An example:

Problem 2. (*Nordic 2020*) Georg has $2n + 1$ cards with one number written on each card. On one card the integer 0 is written, and among the rest of the cards, the integers $k = 1, ..., n$ appear, each twice. Georg wants to place the cards in a row in such a way that the 0-card is in the middle, and for each $k = 1, ..., n$, the two cards with the number k have the distance k (meaning that there are exactly $k-1$ cards between them). For which $1 \leq n \leq 10$ is this possible?

Solution. Consider a colouring where the spaces of the cards are coloured in an alternating pattern of black and white spaces. Without loss of generality, we can say that the two ends of the row, the first and last space, are both black. With this colouring, we can discover new properties of the spaces the cards occupy.



Since the pattern is alternating, there is an equal number of white and black spaces, save for one extra black space. When we have $2n+1$ cards, we need $2n+1$ spaces and of these, $n+1$ will be black spaces and n spaces will be white. This gives us an interesting connection between the parity (is it odd or even?) of n and the parity of the amount of black or white spaces. If n is even, there will be an odd number of black spaces and an even number of white spaces. This gets more interesting when we consider that the 0-card, is always in the middle, letting us know which colour the space 0 occupies has. The middle space is always space number $n+1$. And since the first space is black, all spaces with an odd number are black, while even numbered spaces are white. If n is even, the middle space, $n+1$, will be odd and thus, black. Else, the middle space will be white. We can now include this in our calculations to see which spaces are left to occupy. If n is even, there is an odd number of black spaces, but the 0-card is sure to occupy one of these black spaces in the middle. Meaning that there is an equal number of both black and white spaces left that needs to be occupied (i.e n). If n is odd, the 0-card will occupy a white space, meaning that there are $n-1$ white spaces left to occupy and $n+1$ black spaces left to occupy. These facts regarding the spaces will be used later.

Now we investigate what happens when these pairs of cards are placed upon these spaces. Say we have a pair of cards with the integer k written on them. If we assume that this pair can be placed in a correct manner, we know that if one of these cards occupy the space numbered m, the other cards spaces's number is $m+k$. If k is even, the two spaces will have the same parity and the same colour. However, if k is odd, the two spaces will have different parities, and thus different colours. We now know some things about what happens when these cards are placed in these spaces, and can now start to draw some conclusions.

Firstly, we can prove that it is not possible when $n \equiv 1 \mod (4)$ (i.e n leaves remainder 1 when divided by 4). In this scenario, n is odd, meaning that the non-zero cards need to occupy $n-1$ white spaces and $n+1$ black spaces. However, there is an odd number of pairs of cards with odd integers. If we examine the white spaces that needs to be occupied, we see that there is an even number of white spaces in the the beginning. But since every pair of cards with odd integers need to occupy exactly one white space, there will be an odd number of white spaces left that the pairs with even integers need to occupy. This is; however, impossible. Since a pair of even integers always occupy an even number of white spaces (0 or 2 spaces), the number of white spaces occupied solely by pairs of cards with even integers, will therefore always be even and never odd. We can then conclude that $n = 1$, $n = 5$, $n = 9$ are all impossible.

We can then similarly prove that it's impossible when $n \equiv 2 \mod (4)$. In this scenario, n is even, meaning that the non-zero cards need to occupy n white spaces and n black spaces. There is also an odd number of both pairs with odd integers and pairs with even integers. The proof is now simliar to the one above (i.e the case $n \equiv 1 \mod (4)$). There is an even number of white spaces, an odd number of pairs with odd integers that needs to occupy an odd number of white spaces, leaving an odd number of white spaces for the pairs with even integers. This is of course impossible and we can conclude that all of the cases $n = 2$, $n = 6$, $n = 10$ are therefore also impossible.

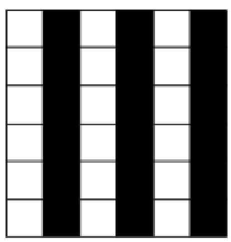
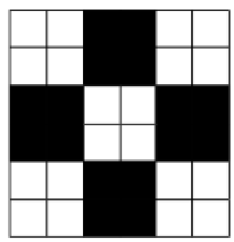
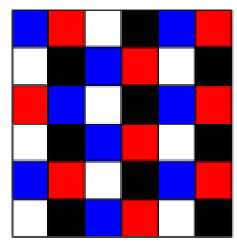
It now simply remains to check whether or not $n = 3, 4, 7, 8$ are possible. As a matter of fact, they are all possible even though this can not be proved by our colouring alone. Just as we stated in the beginning, colouring does not necessarily work as a proof that something exists. We have no choice but to try to construct them ourselves and after putting in diligent effort, we can construct examples for each of these n.

$n = 3$ is possible with the configuration: 2 3 2 0 3 1 1
$n = 4$ is possible with the configuration: 2 4 2 3 0 4 3 1 1
$n = 7$ is possible with the configuration: 3 7 2 3 2 6 4 0 7 5 4 6 1 1 5
$n = 8$ is possible with the configuration: 5 8 2 3 2 5 3 7 0 8 6 4 1 1 7 4 6

Hence, we have proved that these are the only possible n for which $1 \leq n \leq 10$.

$\square$

After presenting these two examples of standard colourings we want to highlight the fact that the possibilities of colourings really are endless, in the sense that you can have essentially as many colours as you'd like and as many patterns you might be able to conjure. Unfortunately, we don't have the space nor the time required to discuss each of these patterns, but here we provide some classical colouring patterns to look out for in the future. These happen to appear surprisingly frequently in colouring problems.

2 Constructing a suitable colouring

Whenever you are unsure of how to find a working colouring it might be better to try to construct an entirely new colouring instead of just trying every standard colouring you know.

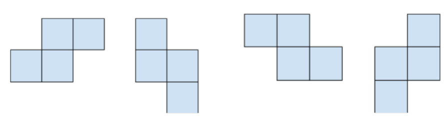
When constructing a new colouring it is important to know what properties you want the colouring to have. So the first thing you want to do is to identify which properties the colouring requires in order to solve the problem.

After finding the properties needed to solve the problem, you should start filling in your colouring. The best way to go about this, is to colour it one colour one element (e.g cell/box on a board) at a time in a deterministic way by letting the properties you require decide for you. Basically, you need to search for elements that can only house a single, specific colour if you want the colouring to have the properties that solves the problem, and in doing so, guarantee that it is correctly placed if such a colouring does exist.

If this process does not yield any contradictions, i.e. it is possible to find a colouring with the desired properties, this process will eventually result in a colouring that should solve your problem.

We'll present an example of this process in action on the following problem.

Problem 3. (*IMOSL 2014*) Construct a tetromino by attaching two 2×1 dominoes along their longer sides such that the midpoint of the longer side of one domino is a corner of the other domino. This construction yields two kinds of tetrominoes with opposite orientations. Let us call them S- and Z-tetrominoes, respectively.



(a) S-tetrominoes (b) Z-tetrominoes

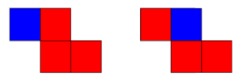
Assume that a lattice polygon P can be tiled with S-tetrominoes. Prove that no matter how we tile P using only S- and Z-tetrominoes, we'll always use an even number of Z-tetrominoes.

Solution. We'll proceed by finding a suitable colouring. First, we need to identify what properties our colouring needs in order to solve the problem. From the problem statement, we can figure out that all S- tetrominoes need to have a property that can only be exchanged for an even number of Z-tetrominoes. The simplest way of achieving this is to have all Z-tetrominoes contain an odd number of each colour we choose to use, whereas the S-tetrominoes should contain an even number of each colour. From this insight, together with the problem statement we deduce that P should contain an even number of each colour, since it is a summation of S-tetrominoes. Furthermore, this means that if P can be tiled, using only S- and Z-tetrominoes, that these Z- tetrominoes altogether need to contain an even number of each colour, which can only happen if there is an even number of Z-tetrominoes. With that said, we know that IF there exists a colouring of the grid such that every placed S- tetromino contains an even number of each colour and every placed Z-tetromino contains an odd number of each colour, the proof is complete.

With these expectations of our colouring established, it becomes much easier to construct a suitable colouring.

We start off by assuming that such a colouring can be achieved with only two colours, red and blue. If it does exist, any Z-tetromino we place should cover one square in the first colour and three squares of the second colour. Since the colours are at this point interchangeable, we can, without loss of generality, say that it covers 1 blue and 3 red squares. Notice that a Z tetromino can cover 1 square of one colour and 3 squares of another colour in two ways, which means that we can't decide which to use deterministically, we have to make a choice, a guess. When using this process it is important to keep track of where in the process of finding a colouring, you make assumptions. This is if you in case do obtain a contradiction, you could gather that one of your assumptions is incorrect and you can backtrack and try out a different assumption instead. We won't have to do this in this particular example, but it is very important to remember when you apply these techniques yourself.

The two options of what our Z-tetrominoes, covering three red and one blue, can look like are:

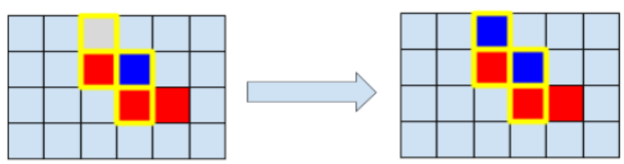


i.e either the Z-tetromino has a blue "outer" square, or a blue "inner" square. Any other coverings can be obtained by simple rotations of the coverings above.

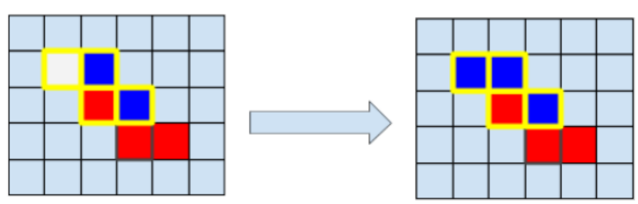
Now, we shall employ our observations and start colouring the board. We assume that the blue square is an "inner" square, i.e the one depicted in the image to the right. It's worth mentioning that this assumption is a pure guess, something that we'll make a note of. You have to start somewhere.

We then proceed by filling in an alternating pattern of "horizontal" Z-tetrominoes and "vertical" S-tetrominoes to construct a "diagonal" across the grid that we know has to be coloured in this way, according to our assumptions none of which has been disproven so far.

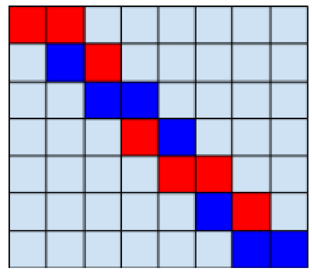
We do this by imagining a vertical S-tetromino that covers three already known squares. The S-tetromino already contains two red, and one blue square. This means that the fourth square must have been a blue square, since that is the only way this S-tetromino would cover an even number of blue squares according to the assumptions we've made so far.



We then proceed to do the same but with an imaginary, horizontal Z-tetromino. Which helps us deduce that the fourth square must be red.

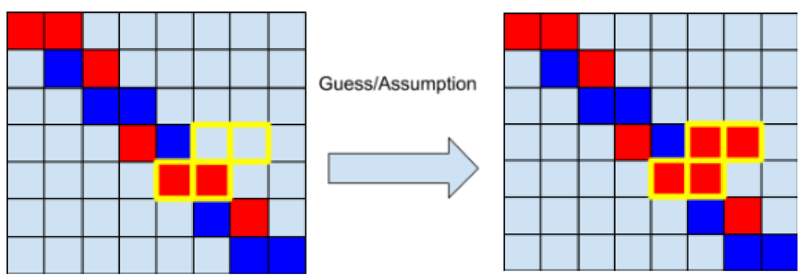


By alternating these two moves we are able to construct a diagonal in both directions

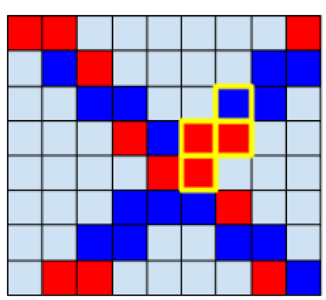


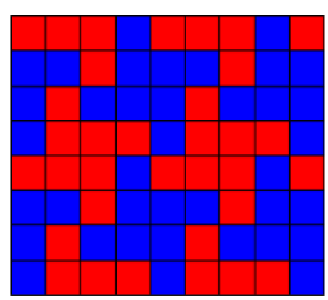
However, in order to fill the whole grid, we'll need to make another assumption. This is because there are no longer any "3 known, 1 unknown" scenarios with the tetrominoes we have. I.e we can't proceed to colour the grid in the same deductive manner as we previously did.

Let's imagine a horizontal S-tetromino. To fulfill the necessary property, the two unknown squares have to be either both red or both blue since the S-tetromino already covers an even number of red squares. In this example, we assume that both are red, as shown in the picture below. Once again, notice that this assumption is a pure guess and if it doesn't work, we would simply return and assume that both are blue instead.



Once we've made this assumption we can use an alternating sequence of horizontal-S-tetrominoes and vertical Z-tetrominoes to construct another perpendicular diagonal, in the same way we did with the first one, i.e using the "3 known, 1 unknown" principle to deduce the colour of one square at the time.





With the help of these "diagonals" and the principle of "three known, one unknown", we may continue to fill in the rest of the grid without making any further assumptions. Upon checking that the colouring satisfies our constraints we obtain that it indeed does and hence the problem statement follows.

Worth mentioning is that when faced with similar questions in a contest situation, presenting the reasoning and motivation behind the colouring IS NOT REQUIRED and should therefore be omitted. All you really need to do is to provide a sketch of your colouring, perhaps explain why it works and then use it to solve your problem.

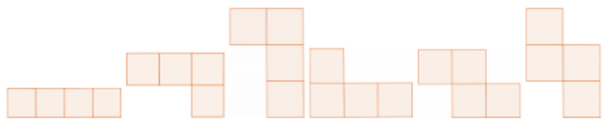
□

3 Extending our colour kit

3.1 Integers

When colouring a board, a colour is a rather arbitrary property than an actual colour, which means that we wouldn't lose any information if integers were to be used instead of colours in a colouring. As a matter of fact, this kind of colouring might even open up more possibilities, revealing previously unknown properties of the setting of our problem. In particular this is partly due to the fact that we may perform operations on our "colours", such as adding them together for instance.

Problem 4. Consider a 6×6-grid. Is it possible to cover the entire grid, using only the pieces provided below?



You are only permitted to translate these pieces onto the board, i.e you are not allowed to rotate or reflect the pieces. Any piece may occur more than once.

Solution. Unsurprisingly, it's impossible! However, before jumping to conclusions and start to work out how to solve the actual problem, it may be a good idea, when faced with tricky questions, like the one above, to actually play along and try to cover the board. The sole reason for this is that in doing so, there is a chance that you might be fortunate enough to accumulate some intuition for the problem, something that might hint at a solution. For instance you may make note of how the board typically looks like after an attempt to cover it? Perhaps there's a pattern that we might be able to distinguish?

One such realization is that of the pieces. How do we encode the property that we may not rotate or reflect the pieces in the board and the pieces? Do the pieces resemble each other in any way? Well, first note that the pieces are very simple in the sense that they aren't turning or twisting in any complicated manner. As a matter of fact it seems as if there is a "path" encoded in the pieces, in the sense that all of these pieces can be completely walked through by using either only leftwards and upwards steps or only rightwards or downwards steps. This suggests that we are looking for an invariant property that might be able to encapsulate this idea.

Consider the following colouring of the board:

2^10	2^9	2^8	2^7	2^6	2^5
2^9	2^8	2^7	2^6	2^5	2^4
2^8	2^7	2^6	2^5	2^4	2^3
2^7	2^6	2^5	2^4	2^3	2^2
2^6	2^5	2^4	2^3	2^2	2^1
2^5	2^4	2^3	2^2	2^1	2^0

Note that any piece with its rightmost-bottom-piece in a box with the number 2^x, will cover a set of integers whose sum is $2^x + 2^{x+1} + 2^{x+2} + 2^{x+3} = 2^x \cdot 15$. It's quite interesting that 15 appears out of seemingly nowhere, which compels us to further investigate how we can exploit this peculiar result. Wishful thinking, it would be quite nice if the sum of all of the integers on the board wasn't divisible by 15, since this would imply that it's impossible to cover the board. And indeed, the sum of all of the integers on the board can be neatly expressed as $(1 + 2^1 + 2^2 + ... + 2^5)^2 = (2^6 - 1)^2 = 63^2$, which isn't divisible by 5 and hence not 15. $\square$

Note that the above problem can indeed be solved by a clever colouring using "ordinary" colours (can you find it?). Nevertheless, it does the job of illustrating how an extension of our colour kit may yield success, and more incredibly, it shows how a tiling problem can be transformed into a number theoretical one. Furthermore, the colouring above provides a deeper understanding of the nature of the problem since it incorporates the idea that the pieces govern an up-left-movement (can you generalize this idea to other pieces?). After all, the reason why the sum of any piece is $2^x \cdot 15$ is due to the fact that moving in an up-left-direction the next square will be double that of the current, something that also holds true for any up-left-walk on the board.

3.2 Complex numbers

So far we've familiarized ourselves with the idea that a colour can be extended to the notion of a weight. Furthermore, as we've seen in the case of integers, there is really nothing that should restrict our domain of colours. And as we shall see, we will draw inspiration from the complex plane and even from the center of mass concept from physics, to further expand our colour kit even more.

Although we assume that the reader is somewhat familiar with the concept of complex numbers and some of their basic properties, we shall yet provide a very brief outline of the topic.

Definition 3.2.1. A complex number is a number of the form $a + b\sqrt{-1}$, where a and b are real numbers. A common notation for $\sqrt{-1}$ is i.

Theorem 3.2.1. *Each complex number has a polar representation $re^{i\theta}$, i.e for an arbitrary complex number $a + bi$ there exists real numbers r and θ such that $a + bi = re^{i\theta}$.*

Finally, before getting into details, recall a famous result due to Euler which asserts that $e^{2\pi i\theta} = 1$ *iff* θ is an integer.

Lemma 3.2.1. *The sum $1 + x + ... + x^{n-1}$ has the closed form $\frac{x^n-1}{x-1}$ if $x \neq 1$.*

Proof. The statement can easily be proven by multiplying the sum with the number $(x - 1)$, i.e

$$(x - 1)(1 + x + ... + x^{n-1}) = x^n - 1$$

from which we conclude that $1 + x + ... + x^{n-1} = \frac{x^n-1}{x-1}$, since $x \neq 1$. $\square$

The sum on the left hand side is typically referred to as a *geometric sum* due to the fact that each term in the sum can be considered to be an incrementation of the previous one by the factor x.

Combining these two results yields a fact that is surprisingly powerful for our purposes, namely

Fact 3.2.1. *Let $m \neq 1$. Then the equality $1 + (e^{\frac{2\pi i}{m}})^1 + (e^{\frac{2\pi i}{m}})^2 + ... + (e^{\frac{2\pi i}{m}})^{n-1} = 0$ is equivalent to $\frac{n}{m}$ being an integer.*

When n, m are integers, we typically refer to the property $\frac{n}{m}$ *being an integer*, as "m divides n" or more densely expressed $m \mid n$.

There is obviously more to the complex numbers and their fruitful properties that we haven't covered, but this is all we really need for the moment.

Before reading the solution to the next problem, we really urge you to try it out yourself first.

Problem 5: It is known that a checkered $m \times n$ board can be covered with $k \times 1$ and $1 \times p$ pieces, such that only one piece covers one square. Show that the board can be covered with only $k \times 1$ or $1 \times p$ pieces as well.

Solution. At first you might be tempted to come up with a system of trying out different colourings of the board. Intuitively, we would like for the colouring to embed some sort of divisibility property between the pieces and the board since if ever were to obtain that $k \mid m$ or $p \mid n$, it would be quite easy to tile the board with only one type of piece. However, it turns out to not be that of an easy task (unless you are very clever of course!).

You might have noticed that one way to go about this is to draw inspiration from fact 3.2.1, since it seems to govern some divisibility property that we might be looking for, and as it turns out it does.

If k or p is 1, the problem is trivial, so let's assume that both are greater than 1. First we would like to orientate the board such that we can refer to its square-pieces as (a, b), i.e the a:th brick from the leftmost edge and b:th brick counting from the bottom edge. In doing so, the square pieces can now be described as (a, b) where $0 \leq a \leq m-1$ and $0 \leq b \leq n-1$ and a and b are both integers. We shall next proceed by colouring each piece of the form (x, y) with the colour $e^{\frac{2\pi i x}{k}} e^{\frac{2\pi i y}{p}}$.

It now follows that the sum of the squares inside any $k \times 1$ and $1 \times p$ brick is 0, due to fact 3.2.1, from which we conclude that the sum of all squares of the board has to be 0. However, noticing that the sum of all the squares on the board can also be expressed as the product

$$\left(1 + (e^{\frac{2\pi i}{k}})^1 + (e^{\frac{2\pi i}{k}})^2 + ... + (e^{\frac{2\pi i}{k}})^{m-1}\right) \left(1 + (e^{\frac{2\pi i}{p}})^1 + (e^{\frac{2\pi i}{p}})^2 + ... + (e^{\frac{2\pi i}{p}})^{n-1}\right)$$

we must have that $k \mid m$ or $p \mid n$ in order for the product to be 0. It now remains to actually cover the board, a trivial task left as an exercise for the reader. □

Notice that the key insight to problem 5, was that the geometric sum of $e^{\frac{2\pi i}{p}}$ is 0 in a $1 \times p$-piece and same goes for the geometric sum of $e^{\frac{2\pi i}{k}}$ in a $k \times 1$-piece. These are the kind of properties we are looking for when extending our colour kit, i.e colours which in their nature govern obscure properties and operations which when properly exploited yields thrilling statements and solutions.

3.3 Center of mass

We shall now move on to our final technique, namely *center of mass*. This technique is arguably the most liberal in the notion of colouring. As you might have noticed, for each extension of our colour kit, we've successively distanced ourselves more and more from the classical concept of colouring to the extent that one might even argue that we aren't even working with colouring anymore. A valid point for instance would be to consider these techniques to be weighting rather than colouring. However, the line distinguishing the two is immensely fuzzy and nevertheless, this degree of abstraction of colours may even help us to better understand what's actually going on when we are working with actual colours for instance or perhaps even weights.

Center of mass is a technique that one usually encounters in olympiad geometry in the shape of *barycentric coordinates*, although its applications can be found in combinatorics, algebra and even analysis. So, what exactly is center of mass? The typical approach one would go about to explain what center of mass is, is by constructing a so called *particle system*.

Definition 3.3.1. Let P_1, P_2, ..., P_n be a finite set of points in the plane. To each point P_j we associate it with a positive real number m_j (also known as its *mass*). Finally define a point O, which we shall call the origin. Then the center of mass of the particle system (P_1, m_1), (P_2, m_2), ..., (P_n, m_n) is a point Z in the plane such that.

$$m_1\overrightarrow{P_1Z} + m_2\overrightarrow{P_2Z} + ... + m_n\overrightarrow{P_nZ} = 0 \tag{1}$$

Note that it isn't immediately obvious that there exists such a point Z. The way we prove that it actually does exist is by constructing it from the given particle system and origin.

Note that we may rewrite (1) as

$$m_1(\overrightarrow{OZ} - \overrightarrow{OP_1}) + m_2(\overrightarrow{OZ} - \overrightarrow{OP_2}) + ... + m_n(\overrightarrow{OZ} - \overrightarrow{OP_1}) = 0$$

which is equivalent to

$$\overrightarrow{OZ}(m_1 + m_2 + ... + m_n) = m_1\overrightarrow{OP_n} + m_2\overrightarrow{OP_n} + \ldots + m_n\overrightarrow{OP_n}$$

Since $m_1, \ldots, m_n$ are positive real numbers we obtain that (1) is equivalent to

$$\overrightarrow{OZ} = \frac{m_1\overrightarrow{OP_n} + m_2\overrightarrow{OP_n} + \ldots + m_n\overrightarrow{OP_n}}{m_1 + m_2 + ... + m_n}$$

Since all of the steps are equivalent we have shown that Z does indeed exist, but not only that, we've also constructed Z.

Fact 3.3.1. *If Z is the center of mass of the particle system (P_1, m_1), (P_2, m_2), ..., (P_n, m_n), then*

$$\overrightarrow{OZ} = \frac{m_1\overrightarrow{OP_n} + m_2\overrightarrow{OP_n} + \ldots + m_n\overrightarrow{OP_n}}{m_1 + m_2 + ... + m_n}$$

for an arbitrary point O in the plane.

Applying fact 3.3.1 we can deduce several important properties of the center of mass.

Fact 3.3.2. *Given a particle system (P_1, m_1), (P_2, m_2), ..., (P_n, m_n), then the center of mass of the system is invariant if we were to replace an arbitrary subset of the particles with the particle (Z', M), where Z' corresponds to the center of mass of the subset of particles and M the sum of their masses.*

Fact 3.3.3. *Given two particles (P, m) and (Q, n) then their center of mass lies on the segment PQ, dividing it in the ratio $n : m$ from P.*

Although center of mass is really interesting in itself, we'll have to conclude our investigation here and return to how we might be able to apply this in typical colouring problems. Consider the following problem.

Problem 6. Can you tile a 6×6 board with 1×4-tiles?

Solution. The problem is obviously no, which can be shown by a simple colouring (can you generalize this statement to arbitrary rectangular boards and $1 \times n$-tiles). However, we would like to apply our acquired knowledge of center of mass coordinates, and as it turns out there is a very elegant and short solution applying this theory.

Assume that there is such a covering. Let each square tile of the board weigh $\frac{1}{4}$ and each 1×4-tile weigh 1 (i.e its mass is distributed according to the square tiles it covers). Then the center of mass of all the pieces has to be the same as that of the board (i.e all of the square tiles). Let the board have one corner in the origin of a standard xy-coordinate system and let its sides be parallel to the axes. Finally let the side length of each of the 36 squares covering the board be 2. Then we obtain that the center of mass of the board is at

$(6,6)$ (something that's easily deduced from fact 3.3.2 and fact 3.3.3).

Now let's have a look at the center of mass of the 1×4-tiles. Note that the center of mass of an arbitrary 1×4-tile on the board has one even and one odd coordinate (also easily deduced from fact 3.3.2 and fact 3.3.3). Denote this set of coordinates by (a_1, b_1), (a_2, b_2), ..., (a_9, b_9) (since there are exactly 9 tiles on the board), and then compute the center of mass of the 1×4-tiles with respect to the origin $(0, 0)$, i.e from fact 3.3.1

$$\frac{(a_1, b_1) + (a_2, b_2) + ... + (a_9, b_9)}{9}$$

or

$$\left(\frac{a_1 + a_2 + ... + a_9}{9}, \frac{b_1 + b_2 + ... + b_9}{9}\right). \tag{2}$$

However, recall that the center of mass of an arbitrary 1×4-tile has one even and one odd cordinate, which means that the numerator of the x-coordinate or y-coordinate in (2) has to be odd. This however, implies that (2) can't be a point with coordinates (6, 6). Since the center of mass of the board was $(6, 6)$, this contradicts the assumption that the board could be covered with 1×4-tiles hence it's impossible to do so. $\square$

Note that the above proof makes use of a more common technique in combinatorics, namely *double counting*. Double counting is a method where one typically seeks to prove the impossibility of a particular statement by computing a certain quantity twice in two different ways. The intended effect of this approach is to obtain a contradiction in the same fashion as in problem 6, i.e computing the center of mass of the board twice, first by immediate computation of the board and the second by the tiles, in order to derive a contradiction.

On a final note on center of mass coordinates. We don't even need for the masses of the particles to be positive or even real for that matter. Furthermore, we could even extend fact 3.3.1 to infinitely many particles. This suggests that we've barely scratched the surface of the potential of center of mass colouring.

4 Conclusion

Once again, it's worthwile mentioning why we choose to present different extensions of our colour kit in section 3. As previously mentioned it illustrates the fact that a colour is a rather arbitrary property and that treating complex numbers as colours for instance may yield elegant and quite intuitive solutions. I.e section 3 shows that one should be open minded when dealing with colours (or any other topic for that matter) in the sense that one shouldn't be stubborn and limit oneself to the standard colours. This might seem controversial, although it really isn't. Extending our domain is actually quite natural and essential in many parts of mathematics. Consider the extension of the integers to the *rational numbers* or the *gaussian integers* (or other *quadratic extensions* for that matter) for instance.

However, this representation of the topic also encourages the reader to explore the realm of colours and that one shouldn't solely rely on previously acquired knowledge and well known methods, but be bold and dare to venture beyond what's standard. Needless to say, this is something that we hint at in section 2, where the reader is encouraged to have a mind of his/her own when dealing with a tricky problem and not only spend time trying to make some particluar method (or well-known colourings in this particular instance) fit.

While section 3 has been dedicated to introducing interesting and exciting ways to use colouring proofs, we want to stress that they are in no way better than the standard colouring that ”only uses regular colours”. We wanted to include them due to their novelty and their nature of pushing the limits of colouring. But it is highly unlikely that any of these methods presented in section 3 would be needed for absolutely any situation. In more than 90 percent of cases, a tiling problem will not need anything else but the standard colouring proof and the difficulty lies in finding the correct colouring.

But if you still wish to learn new methods after mastering the standard colouring technique, we would recommend learning the barycentric coordinates method. It is a fairly lesser known technique for tiling problems, meaning that it often bypasses the intended solution, making problems that are supposed to be solved with complex patterns, easily proved using barycentric coordinates and just crunching the numbers, giving it the nickname ”the bash of tiling problems”, drawing a parallell to ”bashing” technique of solving geometry problems with complex numbers. Of course, it is not completely anagolous to using complex numbers in geometry. For example, while complex numbers do not lose any information about the problem and therefore are logically equivalent to the original question, barycentric coordinates in colouring problems still loses a lot of information meaning that it is not guaranteed to be able to answer the question.

Happy bashing!