Applications of Complex Numbers in Geometry

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1 Introduction

To succeed in mathematics you need a wide array of tools to have a chance of finding one for the problems you encounter. Here we discuss one tool often overlooked in competitive mathematics, the usage of complex numbers to solve geometry problems. Infamous for creating long and unintuitive solutions, complex solutions are often avoided in favour of synthetic solutions but given the right circumstances a complex solution might actually be the best option. Theoretically, all geometry problems can be solved using complex numbers. However, the practicality differs immensely. Sometimes it will just be too messy.

1.1 Forms

Complex numbers can be written using different forms.

Rectangular form Written on the form $z = a + bi$

Polar form Written on the form $z = r(\cos\theta + i\sin\theta)$

Exponential form Written on the form $z = re^{i\theta}$
r is the distance to the point from the origin. θ is the angle the complex number (as a vector) makes with the real axis. Positive angles go counterclockwise and negative angles clockwise. The angle, when speaking of complex numbers, is called the argument.

2 Angles

A frequent application for complex numbers is determining the angle between two lines. The easiest example of this is if you have two complex numbers and you represent both using two vectors. In order to calculate the angle between these two vectors you take the quotient of the numbers. When multiplying two complex numbers you add their arguments. Similarly, when you divide, you subtract the arguments and this difference will be the angle between the lines.

$$z_1 z_2 = r_1 e^{i\theta_1} r_2 e^{i\theta_2} = r_1 r_2 e^{i(\theta_1+\theta_2)}$$

$$\frac{z_1}{z_2} = \frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \frac{r_1}{r_2} e^{i(\theta_1-\theta_2)}$$

Most of the time you will not have a single number which is representative of a line, but rather two numbers which lay on the line, say a and b. The direction of the line will be the equivalent of a vector going from a to b. This vector can be written as $b - a$.

If you have two lines and want to find the angle between them, you may take the quotient. However, if you do not have any actual numbers, then you cannot interpret your answer. This is solved by using a property of conjugate numbers. If your quotient is real, then your answer will be its own conjugate. Take two lines and let a and b lie on line 1 and c and d on line 2. If the two lines are parallel then the following relationship will be true

$$\frac{a-b}{c-d} = \overline{\left(\frac{a-b}{c-d}\right)}$$

$$\frac{a-b}{c-d} = \frac{\overline{a}-\overline{b}}{\overline{c}-\overline{d}}$$

$$\frac{a-b}{\overline{a}-\overline{b}} = \frac{c-d}{\overline{c}-\overline{d}}$$

If two lines are perpendicular to each other then it means that their quotient would be a purely imaginary number. The complex conjugate of an imaginary number is the original number with an inverted sign. Using the same lines as in the previous example, the relationship for perpendicular lines is

$$\frac{a-b}{c-d} = -\overline{\left(\frac{a-b}{c-d}\right)}$$

$$\frac{a-b}{c-d} = -\frac{\overline{a}-\overline{b}}{\overline{c}-\overline{d}}$$

$$\frac{a-b}{\overline{a}-\overline{b}} = -\frac{c-d}{\overline{c}-\overline{d}}$$

There is another useful relation which can be derived from the relationship regarding parallel lines and collinearity. Take three points a, b and c. If a vector going from a to b is parallel to a vector going from a to c then they are collinear. This will result in the relation

$$\frac{a-b}{\overline{a}-\overline{b}} = \frac{a-c}{\overline{a}-\overline{c}}$$

Theorem 2.1. *Two lines AB and CD, with corresponding complex numbers, a,b,c and d are:*

(i) Parallel if and only if $\frac{a-b}{\overline{a}-\overline{b}} = \frac{c-d}{\overline{c}-\overline{d}}$

(ii) Perpendicular if and only if $\frac{a-b}{\overline{a}-\overline{b}} = -\frac{c-d}{\overline{c}-\overline{d}}$

Theorem 2.2. *A, B and C are collinear if and only if* $\frac{a-b}{\overline{a}-\overline{b}} = \frac{a-c}{\overline{a}-\overline{c}}$

3 Circles

Circles often pose a problem for solutions involving complex numbers as most ways of expressing them quickly become unwieldy in more difficult problems. There is however one exception. At the heart of almost every complex solution lies the unit circle. By identifying a prominent circle in the problem statement and letting that circle be the unit circle the solution can be massively simplified using the following properties.

Theorem 3.1. *Given two points a and b on the unit circle it holds that*

(i) $\overline{a} = \frac{1}{a}$

(ii) $\frac{a-b}{\overline{a}-\overline{b}} = -ab$

(iii) The tangents to the unit circle at a and b intersect at $\frac{2ab}{a+b}$

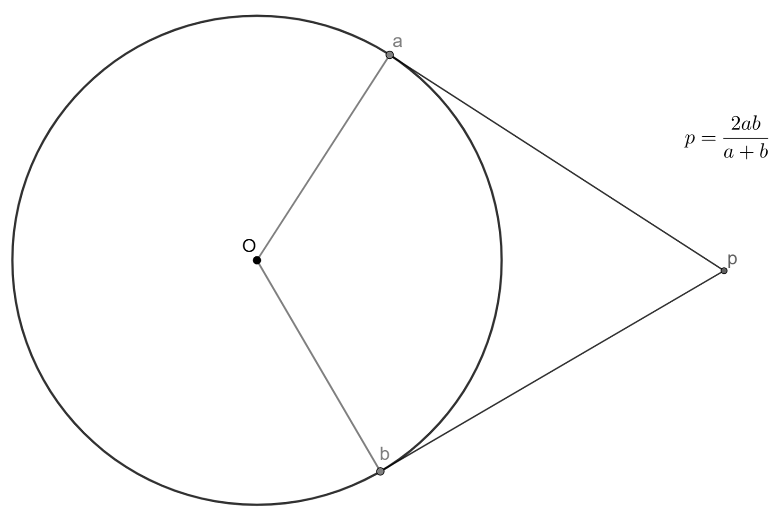

Figure 1: Illustration to Theorem 3.1 (iii). The tangents of two points a and b on the unit circle meet at a point p, where $p = \frac{2ab}{a+b}$

Proof. (i) follows directly from the fact that $\overline{a}a = |a|^2$ ($|a| = 1$ since a lies on the unit circle).□

Using (i), (ii) can now be proven as

$$\frac{a-b}{\overline{a}-\overline{b}} = \frac{a-b}{\frac{1}{a}-\frac{1}{b}} = \frac{(a-b)ab}{b-a} = -ab$$

(iii) is a little harder to prove. AO and PA are perpendicular which means that they can be expressed as

$$\frac{P-a}{\overline{P}-\overline{a}} = -\frac{a}{\overline{a}}$$

$$P-a = -a^2(\overline{P}-\frac{1}{a})$$

$$-\frac{P}{a^2}+\frac{2}{a} = \overline{P}$$

Due to the symmetry of the problem this expression also applies to BO and PB. Then we get the equation

$$-\frac{P}{a^2}+\frac{2}{a} = -\frac{P}{b^2}+\frac{2}{b}$$

$$\frac{P}{b^2}-\frac{P}{a^2} = \frac{2}{b}-\frac{2}{a}$$

$$P = \frac{\frac{2}{b}-\frac{2}{a}}{\frac{1}{b^2}-\frac{1}{a^2}}$$

$$P = \frac{(2a-2b)ab}{a^2-b^2}$$

$$P = \frac{2ab}{a+b}$$

□

Problem 3.1. Given a circumscribed quadrilateral $ABCD$, let M and N be the midpoints of the diagonals AC and BD. If O is the incenter, prove that M, N, and O are collinear.

Proof. Let the inscribed circle of $ABCD$ be the unit circle. According to Theorem 2.1 (iii) it is sufficient to prove that

$$\frac{m-o}{\overline{m}-\overline{o}} = \frac{n-o}{\overline{n}-\overline{o}}$$

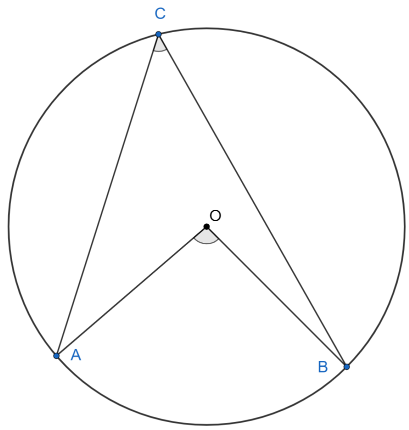

Figure 2: The inscribed angle theorem tells us that $\angle AOC = 2\angle ACB$. It can be proven using complex numbers, see proof to Theorem 3.2.

since O is the origin we need to prove that $\frac{m}{\overline{m}} = \frac{n}{\overline{n}}$. Let p, q, r, s be the points of tangency of the incircle with the sides ab, bc, cd, da respectively. Using Theorem 3.1 (iii) we have

$$m = \frac{a+c}{2} = \frac{ps}{p+s} + \frac{qr}{q+r} = \frac{pqs + prs + pqr + qrs}{(p+s)(q+r)}$$

the conjugate of m will then be $\overline{m} = \frac{p+q+r+s}{(p+s)(q+r)}$. Dividing m by $\overline{m}$ now gives us

$$\frac{m}{\overline{m}} = \frac{pqs + prs + pqr + qrs}{p+q+r+s}$$

Notice that this expression is symmetric in p, q, r and s. Similarly, we will therefore get the same result for $\frac{n}{\overline{n}}$. This gives us that $\frac{m}{\overline{m}} = \frac{n}{\overline{n}}$. $\square$

These properties, unique to the unit circle, makes many problems trivial that would otherwise be infeasable. Complex solutions are therefore best reserved for problems containing no more than one predominant circle. As long as that is true they can produce short and simple proofs. Consider for example a complex proof of the inscribed angle theorem.

Theorem 3.2 (Inscribed Angle Theorem)**.** *Given three points A, B and C on a circle with midpoint O $\angle AOB = 2\angle ACB$.*

Proof. WLOG we assume the circle to be the unit circle and represent the points A, B and C as complex numbers a, b and c respectively. Since a, b and c lie on the unit circle O will be 0. As mentioned earlier, the angle between two lines can be determined by a quotient. We want the angles $\angle AOB$ and $\angle ACB$. However, they are going to be differing by a factor of 2. In order to double the angle of a complex number you square it

$$z^2 = \left(re^{i\theta}\right) = r^2 e^{i2\theta}$$

The problem now is that the magnitudes are squared as well. This is solved by ensuring that the length of all differences are 1. The length from the center of the unit circle to a and b is simply 1 by definition. To get the vector from a to c to have the length 1, you need to divide the vector by its own length. With all of this in mind one gets the expression

$$\left(\frac{\frac{c-a}{|c-a|}}{\frac{c-b}{|c-b|}}\right)^2$$

$$\left(\frac{|c-b|}{|c-a|} \cdot \frac{c-a}{c-b}\right)^2$$

$$\frac{|c-b|^2}{|c-a|^2} \cdot \frac{(c-a)^2}{(c-b)^2}$$

$$\frac{(c-b)(\overline{c}-\overline{b})}{(c-a)(\overline{c}-\overline{a})} \cdot \frac{(c-a)^2}{(c-b)^2}$$

$$\frac{c-a}{\overline{c}-\overline{a}} \cdot \frac{\overline{c}-\overline{b}}{c-b}$$

$$-ca \cdot \frac{1}{-cb}$$

$$\frac{a}{b}$$

$\frac{a}{b}$ is the angle between the A, O and B which we have now shown to be twice that of angle between A, C and B. $\square$

4 Triangles

Equipped to deal with angles and circles the only thing that remains is triangles. Solving geometry problems without a way of expressing triangle centers would be a daunting task. Luckily they can often be expressed using simple expressions.

Perhaps the simplest triangle center to express using complex numbers is the centroid, the point within the triangle where the three medians meet. For a given triangle ABC the centroid can be expressed as $\frac{a+b+c}{3}$.

Expressing the incenter and circumcenter (the center of the inscribed circle and circumscribed circle respectively) in the general case is not as simple. This can however easily be solved by letting the incircle or circumcircle be the unit circle. In the former case it is then often beneficial to express every point in terms of the tangent points instead of the triangle corners. By doing this the incenter or circumcenter simply becomes 0.

Lastly, the orthocenter (the point where the altitudes of a triangle meet) also benefits from having its triangle corners lie on the unit circle. As long as this is the case the orthocenter of a triangle ABC can also be expressed in a simple expression as $a + b + c$.

Proof. Let $h = a+b+c$. Using the perpendicular theorem, we get that AH and BC are perpendicular, if and only

$$\frac{h-a}{b-c} = -\overline{\left(\frac{h-a}{b-c}\right)}$$

$$\frac{b+c}{b-c} = -\frac{\overline{b}+\overline{c}}{\overline{b}-\overline{c}}$$

since b and c are on the unit circle, we can plug in $\overline{b} = \frac{1}{b}$ and similarily $\overline{c} = \frac{1}{c}$. From that we obtain:

$$\frac{b+c}{b-c} = -\frac{\frac{1}{b}+\frac{1}{c}}{\frac{1}{b}-\frac{1}{c}} = -\frac{b+c}{c-b} = \frac{b+c}{b-c}$$

We have proven that AH and BC are perpendicular. By similar reasoning, BH and AC are perpendicular, as well as CH and AB. Thus h is the orthocenter of the triangle □

Using this, problems featuring triangles and triangle centers can swiftly be dealt with. As an example consider the following problem.

Problem 4.1. Let S be the circumcenter and H the orthocenter of ΔABC. Let Q be the point such that S bisects HQ and denote by T_1, T_2, and T_3, respectively, the centroids of ΔBCQ, ΔCAQ and ΔABQ. Prove that $AT_1 = BT_2 = CT_3 = \frac{4}{3}R$, where R denotes the circumradius of ΔABC.

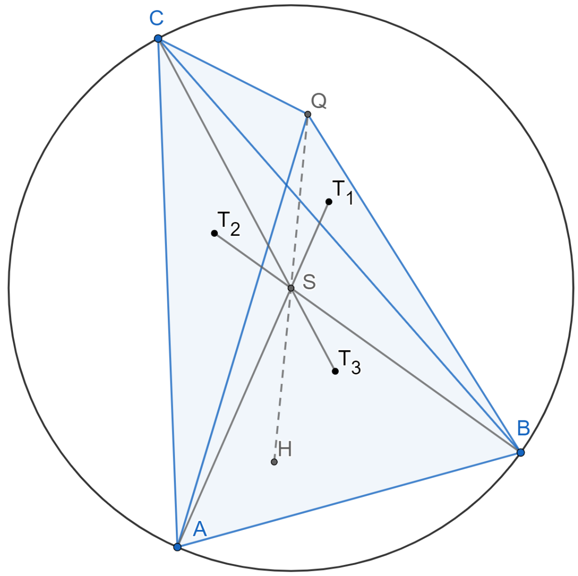

Figure 3: Illustration to problem 4.1. S is the circumcenter of $\triangle ABC$ and also the origin the complex plane. The circumscribed circle of $\triangle ABC$ is the unit circle. H is the orthocenter of $\triangle ABC$ and Q is the reflection of H through S. T_1, T_2 and T_3 are the centroids of ΔBCQ, ΔCAQ and ΔABQ respectively.

Proof. Let the circumcenter of $\triangle ABC$ be the unit circle in the complex plane and a, b and c be complex number representing A, B and C respectively. The orthocenter h can be expressed as $a+b+c$ (theorem). Since s bisects HQ, q is the reflection of h over s, the origin. Thus, $q = -h = -a-b-c$. The centroid of BCQ is $\frac{(q+b+c)}{3}$ which simplifies to $\frac{-a}{3}$. We want to find the distance between a and $\frac{-a}{3}$, i.e.,$|a - (\frac{-a}{3})| = |\frac{4}{3}a| = \frac{4}{3}$. By symmetry this distance is the same as BT_2 and CT_3. □

Problem 4.2. *(IMO 2008)* An acute-angled triangle ABC has orthocenter H. The circle passing through H with center the midpoint of BC intersects the line BC at A_1 and A_2. Similarly, the circle passing through H with center the midpoint of CA intersects the line CA at B_1 and B_2, and the circle passing through H with center the midpoint of AB intersects the line AB at C_1 and C_2. Show that A_1, A_2, B_1, B_2, C_1, C_2 lie on a circle.

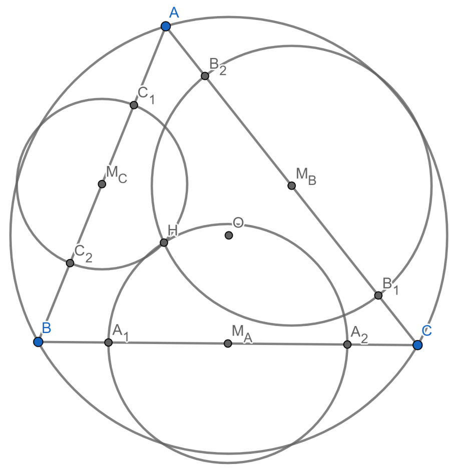

Figure 4: Illustration to problem 4.2. We have also defined M_A, M_B and M_C as the midpoints and O as the circumcenter. We have also drawn the circumscribed circle of the triangle.

Proof. We immediately recognize that if the points were to lie on a circle, its center would have to be the circumcenter, O, because it would lie on the perpendicular bisectors of the sides. Let the circumscribed circle of $\triangle ABC$ be the unit circle. As usual, we will denote the points' complex counterpart as lowercase. Since $\angle OM_BB_2$ is a right angle, we can use the Pythagorean Theorem to get the distance to B_2 from the distance to M_B and the distance from M_B to B_2. Since $m_b = \frac{a+c}{2}$ and the distance from M_B to B_2 is the same as the distance from M_B to H, and that $h = a+b+c$ we can conclude that:

$$|b_2|^2 = |m_b|^2 + |b_2 - m_b|^2 = |\frac{a+c}{2}|^2 + |a+b+c-\frac{a+c}{2}|^2 = |\frac{a+c}{2}|^2 + |b+\frac{a+c}{2}|^2$$

Next we use the fact that the distance to a complex number squared is equal to the complex number times its conjugate. Since a, b and c are on the unit circle, their conjugate will be $\frac{1}{a}$, $\frac{1}{b}$ and $\frac{1}{c}$ respectively.

$$|\frac{a+c}{2}|^2 + |b+\frac{a+c}{2}|^2 = \frac{(a+c)\overline{(a+c)}}{4} + (b+\frac{a+c}{2})\overline{(b+\frac{a+c}{2})}$$

$$= \frac{(a+c)(\frac{1}{a}+\frac{1}{c})}{4} + (b+\frac{a}{2}+\frac{c}{2})(\frac{1}{b}+\frac{1}{2a}+\frac{1}{2c})$$

$$= \frac{1}{4} + \frac{a}{4c} + \frac{c}{4a} + \frac{1}{4} + 1 + \frac{b}{2a} + \frac{b}{2c} + \frac{a}{2b} + \frac{1}{4} + \frac{a}{4c} + \frac{c}{2b} + \frac{c}{4a} + \frac{1}{4}$$

$$= 2 + \frac{a}{2b} + \frac{a}{2c} + \frac{b}{2a} + \frac{b}{2c} + \frac{c}{2a} + \frac{c}{2b}$$

As previously mentioned this is the distance to b_2. There is no need to check the distance to b_1 since we already know it will be the same since the circle's center will be the circumcenter. What's left is doing the last few steps for one of a_1 and a_2, and one of c_1 and c_2 to see that you end up with the same distance, which proves that they lie on a circle. The steps are identical and result is the same answer and is therefore left as an exercise for the reader. However, one can argue that it is not necessary since the only difference in the other two cases will be that either b and c swap places or a and b. Because of the symmetrical nature of the result, it is clear that it would be the same regardless.

□

5 Transformations

In geometry it is important to be able to manipulate figures and points freely. When using complex numbers this becomes easy thanks to a few relations. Translation is easy, it is the same as with cartesian coordinates. You simply add or subtract the number that represents the desired translation. Reflections and rotations are not that difficult either.

The reflection of point z through the origin is, like with cartesian coordinates, $-z$. This is frequently used when a point lies on the unit circle. The reflection of point z_1 through point z_2 will therefore result in $2z_2 - z_1$. Intuitively, it could be seen as taking the vector difference $z_2 - z_1$ and adding z_2. Alternatively, it could be seen as translating the coordinate system so that z_2 gets translated onto 0, reflecting z_1 through 0, and translating back. Reflection of a point Z over line AB can be written as:

$$z' = \frac{(a-b)\overline{z} + \overline{a}b - a\overline{b}}{\overline{a} - \overline{b}}$$

If A and B both lie on the unit circle, it simplifies to:

$$z' = a + b - ab\overline{z}$$

These can be proven by first translating all of the points by taking $-b$ so that b gets translated onto the origin. Since the new line will go through the origin you can simply rotate $z - b$ around the origin by the argument difference of $z - b$ and $a - b$ twice to get the reflection of $z - b$ over the new line. One way to express this new point is:

$$\frac{(a-b)\overline{(z-b)}}{\overline{(a-b)}}$$

The argument of the new point would be $2 * arg(a-b) - arg(z-b)$ which is equal to $2 * (arg(a-b) - arg(z-b)) + arg(z-b)$ as described previously. It will also have the same distance to the origin as $z - b$ since $\frac{a-b}{\overline{(a-b)}}$ will have a distance of 1. Lastly, the new point will have to be translated back by adding b. Simplifying the final expression will result in the general formula above. If A and B were on the unit circle, you can reach the second formula by also using the fact that $b\bar{b} = 1$ and that $\frac{a-b}{\bar{a}-\bar{b}} = -ab$.

Since the projection of a point onto a line is the midpoint of the point and its reflection, these formulas can be used with a simple modification to get the projection. For instance in the second case:

$$z' = \frac{a + b - ab\overline{z} + z}{2}$$

Theorem 5.1. *The reflection of a point Z through the line AB is described by $z' = \frac{(a-b)\overline{z}+\overline{a}b-a\overline{b}}{\overline{a}-\overline{b}}$*

If you wish to rotate a point a certain amount of radians around another point, using cartesian coordinates will be very difficult, but with complex numbers this is very easy. Suppose you wish to rotate a point A φ radians counter-clockwise around another point B. To do this you create the complex number representing the vector from B to A, which is $a - b$. Then this vector is rotated by multiplying it with $e^{i\varphi}$. The following number is your rotated vector starting at the origin. In order to have your rotated vector start from B, you simply add b to the vector.

$$c = (a - b)e^{i\varphi} + b$$

More frequently written as

$$c - b = (a - b)e^{i\varphi}$$

If φ is negative, it will instead rotate clockwise.

Theorem 5.2. *The rotation of A around B φ radians onto C have the relation $c - b = (a - b)e^{i\varphi}$*

6 Lines

A line in the complex plane that goes through two known points z_1 and z_2 can be described as $z = z_1t + z_2(1 - t)$. This is the parametric form with parameter t that can take all real values. Any point p on this line can be described in terms of z_1, z_2 and two real numbers r_1 and r_2 that describe the ratio of the distance from p to z_1 and p to z_2 with $\frac{|r_1|}{|r_2|} = \frac{|p-z_1|}{|p-z_2|}$. The value of this point p is $p = \frac{r_1z_2+r_2z_1}{r_1+r_2}$. If p is on the segment z_1z_2 then r_1 and r_2 are both positive, otherwise one is negative. For example, $p = \frac{z_1+z_2}{2}$ describes the midpoint of segment z_1z_2 and $p = \frac{-z_1+2z_2}{1}$ describes a point outside of segment z_1z_2, twice as far from z_1 as from z_2.

Something that should be avoided when using complex numbers is attempts at calculating the intersection between two lines. There are methods for calculating this but they are very impractical and should be avoided in favor of other methods not involving complex numbers.

7 Practice problems

Problem 7.1. Prove that the midpoints of the altitudes of the traingle are collinear if and only if the triangle is rectangular.

Problem 7.2. Let H_1 and H_2 be feet of perpendiculars from the orthocenter H of the triangle ABC to the bisectors of external and internal angles at the vertex C. Prove that the line H_1H_2 contains the midpoint of the side AB.

Problem 7.3. *(IMO 1998 shortlist)* Let ABC be a triangle such that $\angle ACB = 2\angle ABC$. Let D be the point of the segment BC such that $CD = 2BD$. The segment AD is extended over the point D to the point E for which $AD = DE$. Prove that: $\angle ECB + 180° = 2\angle EBC$.

8 Conclusion

We hope that we have given you some insight into the application of complex numbers in geometric problem solving. If you already knew of the concept but disregarded it as an unviable method we hope to have freed you from this misconception.