Duality in projective geometry. The theorems of Menelaus, Ceva and Desargues

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May 2021

Contents

0 Definitions, Notations and Conventions

The following is good to know in order to understand this text.

- $[ABC]$ denotes the area of ΔABC.
- Three lines are called concurrent if all three intersect at one point.
- Three points are called colinear if there is a line that goes through all of them.
- The *converse* of a theorem is the following: If a theorem is of the form "If A is true, then B is true", then the converse of the theorem states "If B is true, then A is true".

1 The Projective Plane

In this text we will consider the projective plane, which is slightly different from the "normal" Euclidean plane. The Euclidean plane can be extended into the projective plane by defining a so-called *Line at infinity.* It is defined in the following way:

Let l_∞ be a set of points. These are not points in the Euclidean plane, but satisfy the following:

- All lines l_0 in the Euclidean plane intersect l_∞ exactly once.
- If l_1, l_2 are parallel lines in the Euclidean plane they intersect on l_∞ at the same point.
- If l_1, l_2 are lines not parallel in the Euclidean plane, they intersect l_∞ at different points.
- All points in l_∞ lie on a line

You can think of this line as being infinitely far away in all directions, going "around" the Euclidean plane. There is a point at infinity for every "direction", and the union of all these points is the line at infinity. Note that since every line only intersects the line at infinity once, this intuition about "direction" requires you to consider a line as having only one "direction" rather than two.

The main advantage of adding this line, particularly in this text, is the fact that parallel lines no longer exist the way they did in the Euclidean plane. Before, two parallel lines didn't intersect. With the line at infinity, they do intersect (and in fact are concurrent with the line at infinity).
This leads us to the following observation about the projective plane:

- Every pair of two points coincides with exactly one line.
- Every pair of two lines coincides with exactly one point.

This motivates the principle of projective duality.

Proposition 1 (principle of projective duality)**.** *If a true statement about the projective plane is only about lines, points, and which points lie on which lines, then replacing every incidence of the word "point" with the word "line" (as well as making other grammatical changes) yields another true statement.*

The new statement is called the "dual" of the original statement. This is of course a very informal definition. Perhaps more accurate is to say:

Theorem 1.1. *There exists a bijective transformation f from the set of all points and all lines in the projective plane onto itself, which satisfies for all points p and lines l:*

1. $f(p)$ *is a line*
2. $f(l)$ *is a point*
3. $f(f(p)) = p$
4. $f(f(l)) = l$
5. $f(l)$ *lies on* $f(p) \iff p$ *lies on* l

If you do not know what bijective means, reading about it would be a good idea. We omit the definition here for the sake of brevity.

We will prove this theorem by constructing an example in the next section.

This transformation takes any configuration to its dual. Also important is that the dual of the dual of a configuration is the configuration itself (this is also called f being an *involution*).

Notably, he dual of three concurrent lines is three colinear points. The dual of three colinear points is three concurrent lines.

2 Example of an involution with desired properties

We will now construct an example of a transformation which has the desired properties.

Definition 2.1. *Let a transformation f from the set of all points and all lines in the projective plane onto itself be defined as follows:*

Let O be an arbitrary point which could be called the origin.

For any point P, $f(P)$ is defined as the line perpendicular to the line OP through P', where P' is a point on line OP such that $OP \cdot OP' = 1$. If P is on the line at infinity we instead define P' so that $OP' = 0$, which means that the line $f(P)$ is the line perpendicular to OP going through O. $f(O)$ is defined as the line at infinity.

For a line l, $f(l)$ is defined as the point Q' such that $OQ \cdot OQ' = 1$, where Q is the point on l closest to O. If $OQ = 0$ we instead define Q' as the intersection of the extension of l and the line at infinity. You can think of this as "$0 \cdot \infty = 1$", but please note that this is incredibly informal. If l is the line at infinity $f(l)$ is O.

This transformation is very closely related to inversion and poles and polars. We omit specific mention of these concepts here for brevity, but they are very interesting and useful in general, and we recommend reading about them if you want to learn more.

It is clear that f of any line is a point and f of any point is a line. We can also see that for every line l we have $f(f(l)) = l$ and that for every point p, and that $f(f(p)) = p$. We will now show that our fifth desired property holds.

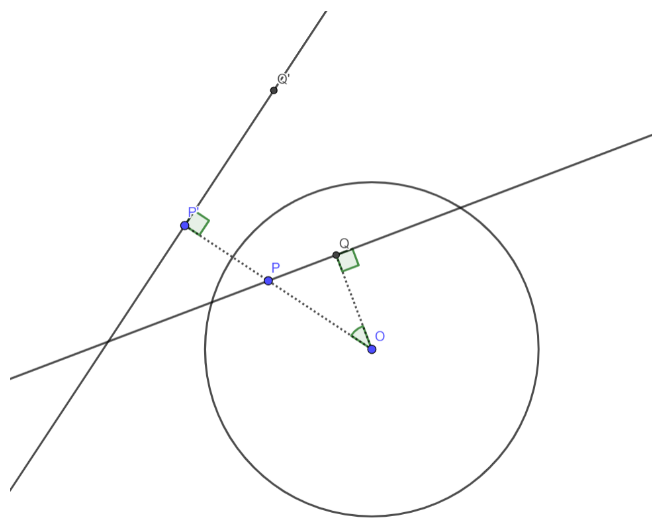

Figure 1: Proving desired property 5.

Theorem 2.1 (Fifth desired property)**.** *For our bijective transformation f, which we defined earlier, this holds:*

$f(l)$ lies on $f(P) \iff P$ lies on l

Proof. The point P lies on the line l.

Let the origin be called O. Let the point P' closest to the origin on $f(P)$. Denote by Q the point on l closest to the origin and let $f(l) = Q'$.

In the general case, assuming that P is not on the line at infinity and not on O and that O is not on l and that l is not the line at infinity, we have from the definition of f that $OP \cdot OP' = 1$ and that $OQ \cdot OQ' = 1$. Let the angle between OP and OQ be α. Since $OQ = OP \cdot \cos\alpha$ we have $OP \cdot \cos\alpha \cdot OQ' = 1$. And since $OP \cdot OP' = 1$ we have $\cos\alpha \cdot OQ' = OP'$. Which means the point $Q' = f(l)$ must lie on the line $f(P)$.

If the point P is the point O we know by definition that $f(P)$ is the line at infinity. We also know that l must contain $P = O$ which means that $f(l)$ is a point on the line at infinity.

If the point P is on the line at infinity, we have from the definition that $f(P)$ is a line through O which is normal to the line OP. If l is the line at infinity, we know that $f(l) = O$ and O is on $f(P)$. Now assume l is not the line at infinity. $f(l)$ is a point on the line OQ by definition. OQ is normal to l and $f(P)$is normal to OP. l and $f(P)$ are parallel and meet at infinity and so OQ and $f(P)$ must also be parallel and meet at infinity. Since they both go through O they are the same line and $f(l)$ lies on $f(P)$ in this case as well.

□

3 Ceva's Theorem

Theorem 3.1 (Ceva's Theorem)**.** *Let ΔABC be a triangle in the Euclidean plane. Let X, Y, Z be points on segments BC, CA, and AB respectively. AX, BY, CZ are concurrent if and only if*

$$\frac{BX}{XC} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB} = 1$$

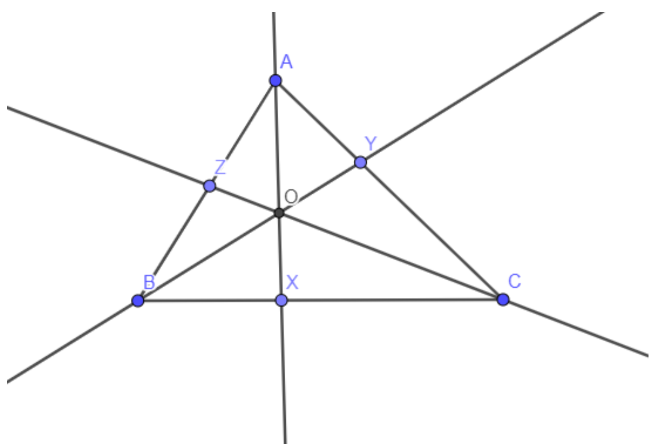

Figure 2: Ceva's Theorem

Proof. First we assume that the points meet at a point O, and show that the ratio condition holds. ΔBAX and ΔCAX share a side, and ΔBOX and ΔXOC also share a side. We get

$$\frac{BX}{XC} = \frac{[BAX]}{[CAX]} = \frac{[BOX]}{[COX]}$$

Since if $\frac{a}{b} = \frac{c}{d}$, $\frac{a}{b} = \frac{a+c}{b+d}$ (Verify this yourself), we get

$$\frac{BX}{XC} = \frac{[BAX] - [BOX]}{[XAC] - [XOC]} = \frac{[BAO]}{[CAO]}$$

Similarly, we get

$$\frac{CY}{YA} = \frac{[CBO]}{[ABO]} \text{ and } \frac{AZ}{ZB} = \frac{[ACO]}{[BCO]}$$

Multiplying, we get

$$\frac{BX \cdot CY \cdot AZ}{XC \cdot YA \cdot ZB} = \frac{[BAO] \cdot [CBO] \cdot [ACO]}{[CAO] \cdot [ABO] \cdot [BCO]} = 1$$

Now the other direction needs to be proved, i.e if the ratio condition holds then the lines are concurrent. Let X, Y and Z denote the same points as before. Now assume X' is a point on segment BC such that

$$\frac{BX'}{X'C} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB} = 1$$

We aim to show that $X = X'$. First note $\frac{BX}{XC} = \frac{BX'}{X'C}$ so if $BX > BX'$, then $XC < X'C$, so $\frac{BX}{XC} > \frac{BX'}{X'C}$, a contradiction. By the same argument $BX < BX'$ cannot be true. So $BX = BX'$ which means $X = X'$ which completes the proof $\square$

4 Menelaus' theorem

In Menelaus' theorem, the notion of directed length ratios is used. For points A,B and C, the directed ratio $\frac{AB}{BC}$ Is positive if B is between A and C, and negative otherwise.

Theorem 4.1 (Menelaus' theorem)**.** *Let ΔABC be a triangle in the Euclidean plane, Let X, Y, Z be three points (none of which are A, B or C), which lie on lines BC, CA, and AB respectively. X, Y and Z are colinear if and only if*

$$\frac{BX}{XC} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB} = -1$$

Where the ratios are directed

Note the fact that the configuration in Menelaus' theorem is the projective dual of Ceva's theorem. Also note how similar the statements are to each other. However, this does not mean we can just refer to the principle of projective duality and consider Menelaus' theorem to be proved, since Ceva's theorem uses line segment lengths in its statement (the principle of projective duality as we have formulated it only applies on theorems which only refer to which

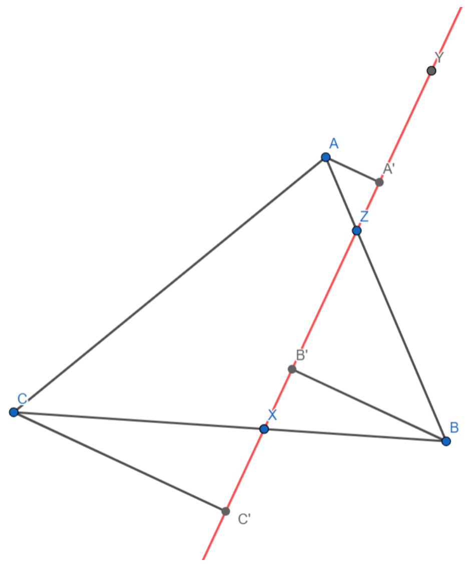

Figure 3: Menelaus' Theorem

points lie on which lines). Later on in the text we present theorems that can be proved just by referring to duality, but for now we will prove Menelaus' theorem directly.

Proof. We prove that if X,Y,Z lie on a line the ratio condition holds.

Since a line not passing through a vertex intersects none or two of the sides of a triangle, either one or three of the ratios will be negative. Hence the product will be negative, and specifically the product of the directed fractions will be exactly -1 times the product of the undirected counterparts.

It therefore suffices to show that

$$\frac{BX}{XC} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB} = 1$$

where the ratios are undirected.

Let A', B' and C' be the feet of the perpendiculars to line XYZ from A, B, C respectively. Then

$$\Delta BB'X \sim CC'X,\ \Delta CC'Y \sim \Delta AA'Y, \text{ and } \Delta AA'Z \sim \Delta BB'Z$$

(verify these yourself)

So we have

$$\frac{AZ}{ZB} = \frac{AA'}{BB'},\ \frac{BX}{XC} = \frac{BB'}{CC'}, \text{ and } \frac{CY}{YA} = \frac{CC'}{AA'}$$

Multiplying these, we get

$$\frac{BX}{XC} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB} = \frac{AA' \cdot BB' \cdot CC'}{BB' \cdot CC' \cdot AA'} = 1$$

as desired. □

Exercise 1. This proof doesn't actually prove Menelaus theorem as stated, for the proof to be complete we still need to show that if the ratio condition is true, then X, Y, Z are colinear. This is similar to an analogous part of the proof of Ceva's theorem, and is handled in a very similar way. Show your understanding of the proof of Ceva's theorem by completing the proof of Menelaus' theorem

Excercise 2. It is in fact possible to restate Ceva's theorem using directed ratios. In Menelaus' theorem, using directed ratios changed the product from a "1" to "-1". Is the same thing true for Ceva's theorem? Why or why not? Also prove the version of Ceva's theorem that uses directed ratios.

Excercise 3. Prove Ceva's theorem using Menelaus' Theorem. (This does not require projective duality)

Excercise 4.

- Prove that the three medians of a triangle intersect using Ceva's theorem
- Prove that the three angle bisectors of a triangle intersect using Ceva's theorem
- Prove that the three altitudes of a triangle intersect using Ceva's theorem

5 Desargues' Theorem

Theorem 5.1. *Let ABC and $A'B'C'$ be two triangles in the projective plane. let P, Q, R be the intersections of BC and $B'C'$, AC and $A'C'$, AB and $A'B'$ respectively. P, Q, R are colinear if and only if AA', BB' and CC' are concurrent.*

Interesting thing to note: There is a beautiful and simple proof of this theorem which utilizes three dimensions. This theorem actually turns out to be an example of one that is easier to prove in three dimensions than two. We do however stay in the two dimensional projective plane in this text in order to better demonstrate usage of Menelaus' theorem and projective duality. If you are curious about the three-dimensional proof, it can easily be found online.

Proof. This theorem is interesting, because the dual of the configuration is the configuration itself. How this helps us will become apparent as the proof progresses. Let us introduce two statements:

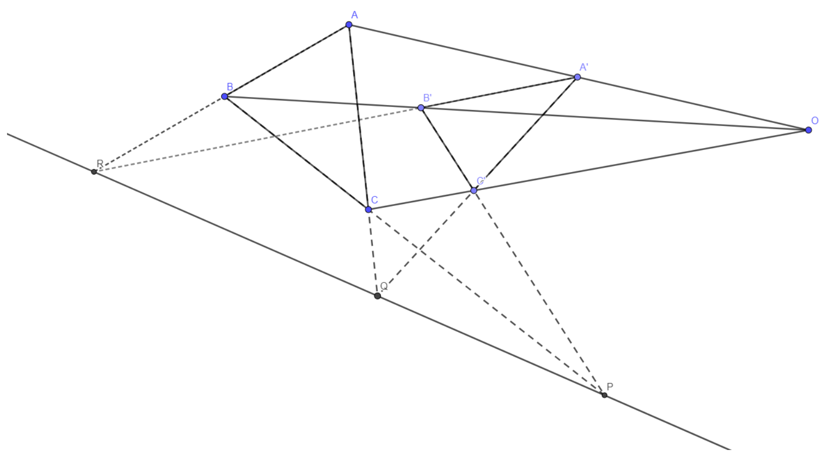

Figure 4: Desargues' Theorem

Statement 1. *Consider the lines of the sides of ΔABC and $\Delta A'B'C'$, then take the points that are incident with each pair of corresponding lines. These points are incident with a single line.*

If we replace every instance of the word "line" with "point", and vice versa, we get the following

Statement 2. *Consider the **points** of ΔABC and $\Delta A'B'C'$, then take the **lines** that are incident with each pair of corresponding points. These **lines** are incident with a single **point**.*

In the notation in the theorem, Statement 1 is the same as "P, Q, R are colinear" and Statement 2 is the same as "AA', BB', CC' are concurrent." First, we will prove Statement 2 $\implies$ Statement 1.

In order to prove this we want to use Menelaus' theorem. However, this leads to a problem. Menelaus' theorem uses length ratios, which are hard to define when some of the points involved are at infinity (in particular, the case where one of the lines of the triangle in Menelaus' theorem is the line at infinity becomes difficult to interpret). However, we have a way to deal with this. This is where the principle of projective duality, and in particular the transformation f we have introduced, is useful.

Assume we have a configuration as described in the statement of Desargues' theorem, where one or more points described lie on the line at infinity, or one of the lines described is the line at infinity. Also assume, for the sake of contradiction, that Statement 2 is true but Statement 1 is false. Now apply the transformation f on the plane twice, with different choices of origin each time. When choosing the origin for the second transformation, choose a point not on any of the points or lines in the dual configuration. This will, by the definition of f, guarantee that none of the points or lines at the following configuration are at infinity. The result is a configuration as described in Desargues' theorem, but with no lines or points at infinity. However, by property 5 of f, for any point p and line l

$$f_2(f_1(p)) \text{ lies on } f_2(f_1(l)) \iff f_1(l) \text{ lies on} f_1(p) \iff p \text{ lies on } l$$

Therefore, for this new configuration, Statement 2 is true but Statement 1 is false. It therefore suffices to show the case where no points in the configuration are at infinity. If a counterexample existed, one would exist with none of the points at infinity (using f twice constructed such a case).

We are therefore safe to use Menelaus' theorem and segment lengths in general. Use Menelaus' theorem on the following triangles and lines:

- ΔBCO and line $B'C'P$
- ΔCAO and line $C'A'Q$
- ΔABO and line $A'B'R$

Following equalities follow:

- $$\frac{BP}{PC} \cdot \frac{CC'}{C'O} \cdot \frac{OB'}{B'B} = -1$$
- $$\frac{CQ}{QA} \cdot \frac{AA'}{A'O} \cdot \frac{OC'}{C'C} = -1$$
- $$\frac{AR}{RB} \cdot \frac{BB'}{B'O} \cdot \frac{OA'}{A'A} = -1$$

Where the ratios are directed. Multiplying these, we get

$$(\frac{CC'}{C'O} \cdot \frac{OC'}{C'C}) \cdot (\frac{AA'}{A'O} \cdot \frac{OA'}{A'A}) \cdot (\frac{BB'}{B'O} \cdot \frac{OB'}{B'B}) \cdot \frac{BP}{PC} \cdot \frac{CQ}{QA} \cdot \frac{AR}{RB} = -1$$

Since for any points X,Y,Z, $\frac{XY}{YZ} \cdot \frac{ZY}{YX} = 1$ where the ratios are directed, we get:

$$\frac{BP}{PC} \cdot \frac{CQ}{QA} \cdot \frac{AR}{RB} = -1$$

So PQR are colinear by Menelaus' Theorem.

As for the converse, i.e PQR colinear $\implies AA', BB', CC'$ concurrent, we get to use projective duality again.

We have already shown that Statement 2 $\implies$ Statement 1.

Assume for the sake of a contradiction that we have ΔABC and $\Delta A'B'C'$ such that Statement 1 is true, but Statement 2 isn't. Choose a arbitrary point to be the origin and apply the involution described in section 2. The resulting is a configuration where Statement 2 is true but Statement 1 isn't. This contradicts what we have proved. Hence Statement 1 $\implies$ Statement 2, and the two statements are equivalent. This completes the proof. □

Note that "Statement 2 $\implies$ Statement 1" and "Statement 1 $\implies$ Statement 2". Are equivalent because of projective duality. These statements are said to be the *duals* of each other

6 Other examples

Pascal's theorem and Brianchon's theorem are two examples of theorems that are duals of each other. If you want to learn of more examples where this concept is useful, those are a good place to start.