Eulers's theorem for polyhedra

Erik Hedin[1], Alexander Jönsson[2], Alexander Nord,
Sigrid Westergren, Richard Xie

Katedralskolan, Lund

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Contents

[1] Gymnasieskolan Spyken
[2] Malmö Borgarskolan

1. Introduction

Euler's formula for polyhedra is a simple formula that is not broadly applicable, however, when used correctly, it turns out to be very efficient. It can be further generalised, but requires a lot more knowledge, thus in this paper, we will focus more on the application of the formula instead of further studies. Nonetheless, there will still be a section for the generalized formula in the end for interested readers.

1.1 Definition for polyhedra

There exists no universal definition of a polyhedron. Several definitions with alternating levels of rigorousness have been given, but there has never been any agreement when it comes to which of these definitions should be used. Nevertheless it is agreed that all polyhedra can be described by vertices, edges and faces. The faces are flat polygons, which are connected to each other through shared sides, forming edges of the polyhedron, and shared corners, forming vertices of the polyhedron. Convex polyhedra are more easily defined. The geometrical interpretation is that a convex polyhedron is a polyhedron where the line segment between any two points on the surface of the polyhedron lies completely within or on the surface of the polyhedron.

A platonic solid is a convex regular polyhedron. Being regular means that all faces are congruent (having the same size and shape). It also means that all faces are regular polygons and that the same number of faces connect at each vertex. As we will later see, only five such polyhedra exist.

1.2 Notation

By convention this paper will use the letters V, E, F to denote the sets of all vertices, edges and faces respectively of some polyhedron. The number of vertices, edges or faces on the polyhedron can then be denoted as $|V|, |E|, |F|$ respectively. Furthermore the *degree* of any vertex or face is the number of adjacent edges to that vertex or face, and this is denoted as $\deg(v)$ for $v \in V$ or $\deg(f)$ for $f \in F$.

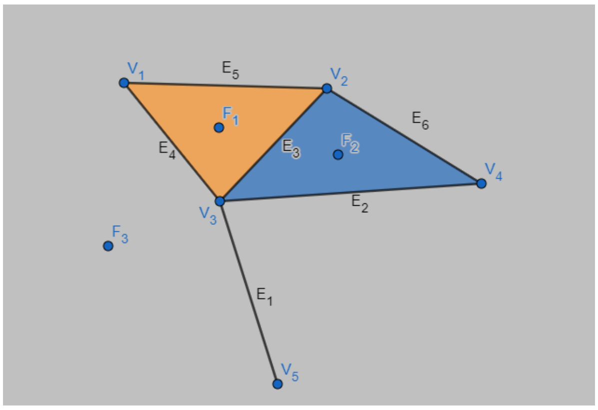
2. Euler's formula for polyhedra

2.1 Statement

Euler's formula for polyhedra states that for any convex polyhedron the following relation holds true:

$$|V| - |E| + |F| = 2,$$

where |V| is the number of vertices, |E| the number of edges and |F| the number of faces of the convex polyhedron. However, every convex polyhedron can also be represented as a planar graph and thereby Euler's formula can also be applied in graph theory for so called planar graphs, and is invariant for every connected planar graph.



Let us apply Euler's formula to a graph to see how it functions. Firstly we notice that it is connected (There is a path from any vertex to any other vertex in the graph) and that no edges intersect (it is planar). As a result Euler's formula should be true. In this graph there are five vertices (V_1, V_2, …, V_5), six edges (E_1, E_2, …, E_6) and three faces (F_1, F_2, F_3). Notice that also the outer area is considered a face. Plugging in these values gives us:

$$5 - 6 + 3 = 2$$

which is indeed correct!

2.2 Geometric proof

The idea of this geometric proof is to start with any polyhedron and then initiate a process of deconstruction, while maintaining knowledge about the number of edges, vertices and faces. Eventually we will reach a simple figure, which we know the properties of. From this we work backwards and derive the facts about our original polyhedron.

First, we draw diagonals in such a way that we get a new polyhedron where all sides are triangles. Notice that everytime we draw a diagonal, the number of vertices does not change, but the number of faces and edges both increases by one. Thus, the following equality holds:

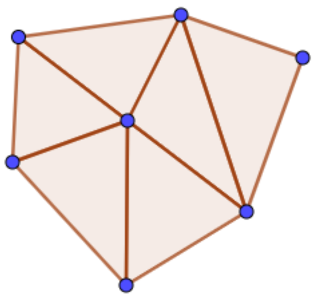
$$|V'| - |E'| + |F'| = |V| - |E| + |F|$$

Where V, E and F are the number of vertices, edges and faces respectively in the original polyhedron, and V', E' and F' are the number of vertices, edges and faces respectively in the new polyhedron with triangle sides.

Now we remove one side from the new polyhedron. We get a figure with the same number of vertices and edges as the polyhedron, but the number of faces reduces by one (since we removed one side). We then continue by "flattening out" the remainder of the figure. We obtain a flat surface without changing the number of vertices, edges and faces. Hence, the following equality holds:

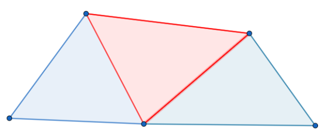
$$|V''| - |E''| + |F''| = |V'| - |E'| + |F'| - 1$$

Where V'', E'' and F'' are the number of vertices, edges and faces in the new figure.

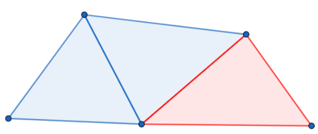


The flattened out figure

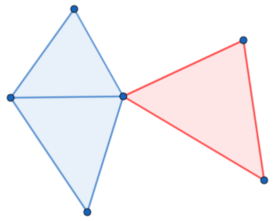
We continue by removing triangles alongside the perimeter of the figure. There are three cases.



Case 1: One edge and one face are removed



Case 2: Two edges, one face and one vertex are removed



Case 3: Three edges, two vertices, and one face are removed

It is easy to check that in all three cases,

$$|V| - |E| + |F|$$

remains the same. By continuing this process we will eventually wind up with only one triangle. In this triangle

$$|V| - |E| + |F| = 3 - 3 + 1 = 1$$

and due to the fact that removing a triangle does not change the value of the expression, the flattened out surface we started with will also have:

$$|V''| - |E''| + |F''| = 1$$

Thus,

$$|V'| - |E'| + |F'| = 1 + |V''| - |E''| + |F''| = 2$$

Where V’, E’ and F’ are the number of vertices, edges and faces respectively in our original polyhedron with triangle sides. This is exactly what we wanted to prove, hence we are done.

2.3 Inductive Proof

To prove the formula we look at two cases, namely a graph with no cycles and thereafter a graph with at least one cycle. These two cases cover all possible graphs.

Proof for Euler’s characteristic formula for trees:
A tree is a graph containing no cycles. We will prove that Euler’s formula is legitimate for all trees by induction on V.

Base case:
Start with the base case, where $|V| = 1$. This graph by definition consists of only one vertex. Therefore no edges can exist and $|F| = 1$ because there are no edges to divide the plane into several faces. This gives us:

$$|V| - |E| + |F| = 1 - 0 + 1 = 2$$

Which is correct.

Induction step:
Assume that the statement holds true for all graphs with $|V| \leq n$
Now assume that we have a tree T, with

$$|V| = n + 1$$

Hence we need to show that $|E| = n$, since $|F| = 1$. To do this we find a vertex with degree one. We can always find this vertex by starting at any vertex and going into any direction never returning, because there are no cycles, we will never return to a previously visited vertex. Consequently, as there are a finite number of vertices we will eventually run into a “dead end”. This vertex must have degree one. If we remove this vertex and the edge from it we end up with a graph containing n vertices and n-1 edges according to our induction hypothesis. As a result, T has n edges and this concludes our inductive proof.

Proof for Euler’s characteristic formula:
We will show this by induction on |E|. The base case is executed in exactly the same way as for trees.

Induction step:
Assume that it holds for all graphs where $|E| \leq n$. Let Q be a connected planar graph with no intersections and $n + 1$ edges. We now have two cases.
Case 1, Q has no cycles: In this case, the graph is a tree and we have already proved this case.

Case 2, Q has at least one cycle: Assume that Q has v vertices and f faces. Choose a cycle and remove one of its edges. We get a new graph P, which is the same graph as Q, except for one edge missing, which means P has n edges, $f - 1$ faces and v vertices. It is obvious that P is a connected planar graph with no intersections, thus Euler’s formula holds, and we get:

$$|V| - |E| + |F| = v - n + (f - 1) = 2$$

Rearranging:

$$|V| - |E| + |F| = v - (n + 1) + f = v - n + (f - 1) = 2$$

Since Q has v vertices, f faces and $n + 1$ edges, we have proven that Euler’s formula also holds for Q, and we are done.

Remark. Every edge is connected to two vertices, thus.

$$\sum_{v \in V} \deg(v) = 2|E|.$$

Also, in polyhedra, $\deg(v) \geq 3$. Therefore: $2|E| = \sum_{v \in V} \deg(v) \geq 3|V|$
$\Rightarrow 2|E| \geq 3|V|$. Using Euler's Characteristic formula and rearranging:
$|V| + |F| = |E| + 2 \geq \frac{3|V|}{2} + 2 \Rightarrow |F| \geq \frac{|V|}{2} + 2$.
Furthermore, every graph has a dual where vertices and faces swap. So do polyhedra. Therefore the sum over the number of sides of all faces of the polyhedron is also 2|E|.

3. Application and other related problems

In section three we will discuss a range of problems where Euler's formula is helpful.

Example 3.1. *There are three houses A, B and C. There are also three utilities: water, gas and electricity all originating from different points. Is it possible to connect all utilities to all houses without any of the lines intersecting?*

Solution: We will prove that it is impossible using proof by contradiction. Imagine a scenario where all requirements are fulfilled. Meaning that we have a planar graph connecting all utilities to all houses. If we inspect the properties of this graph we realize that Euler's formula holds true; we have no intersections and clearly we also have a connected graph. Hence:

$$|V| - |E| + |F| = 2.$$

Furthermore,
$|V| = 6\,(3\, houses + 3\, utilities),\ \ |E| = 9\,(3\, edges\ from\ each\ house\ to\ each\ utility)$
Since Euler's formula is to hold true: $|F| = 5$

We know that the boundary of every face is a closed cycle of edges. We notice that edges going between two houses or between two utilities are unnecessary, because if there is a solution containing any such lines, the situation without any such lines would obviously also suffice as a solution, since all houses would still be connected to all utilities without these lines. Therefore we can look at the case where no such lines exist. This means that every face is bounded by at least four edges. Five faces as required by Euler's formula would necessitate $5 \times 4 = 20$ edges. However, every edge is a member of the boundary of two faces. Consequently at least $20/2 = 10$ edges are required. We have reached a contradiction, since the number of edges cannot equal nine and ten at the same time.

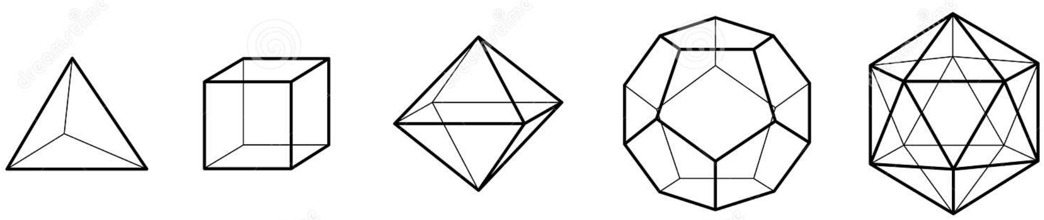
Example 3.2.

- ***a.*** *Show that the only platonic solid with square sides is the cube.*
- ***b.*** *Show that the only platonic solid with pentagon sides is the dodecahedron.*

<u>*Proof:*</u> For both squares and pentagons, the only number of regular polygons we can fit at each vertex of the polyhedron is three, because a vertex cannot have only two polygons or more than four squares or regular pentagons since:
$360° \leq 4 \cdot 90° \leq 4 \cdot 108°$.
For each square we have four edges and four vertices, shared by two resp. three squares. Therefore $|V| = 4|F|/3$, $|E| = 4|F|/2$. Plugging these results into Euler's formula and solving for |F| yields $|F| = 6$, which is the cube.
For each pentagon we have five edges and five vertices, shared by two resp. three pentagons. Therefore $|V| = 5|F|/3$, $|E| = 5|F|/2$. Again, plugging these results into Euler's formula and solving for |F| yields $|F| = 12$, which is the dodecahedron.

Example 3.3. *Show that the tetrahedron, cube, octahedron, dodecahedron and icosahedron are the only platonic solids.*

<u>*Proof:*</u> Let the platonic solid be made up of regular *b*-gons of which *a* come together at each vertex. Note a, b ≥ 3 by definition. Then $2|E| = \sum_{v \in V} \deg(v) = a \cdot |V|$, and $2|E| = \sum_{f \in F} \deg(f) = b \cdot |F|$. Rearranging these equations yield: $|V| = \frac{2|E|}{a}$, and $|F| = \frac{2|E|}{b}$. Plugging these results ino Euler's formula and rearranging yields:
$\frac{2|E|}{a} - |E| + \frac{2|E|}{b} = 2 \Rightarrow \frac{1}{a} + \frac{1}{b} = \frac{1}{2} + \frac{1}{e}$, and hence $\frac{1}{a} + \frac{1}{b} > \frac{1}{2}$.
Now we notice that if both $a \geq 5$ and $b \geq 5$ then the inequality cannot be satisfied because the LHS then becomes smaller than ½. Hence either $a < 5$ or $b < 5$ and we can divide into cases and check that the only pairs (a, b) satisfying the inequality are: (3, 3), (4, 3), (3, 4), (5, 3), (3, 5) corresponding to the tetrahedron, cube, octahedron, dodecahedron and icosahedron respectively:



Example 3.4. *Does there exist a polyhedron with an odd number of faces, each with an odd number of edges?*

<u>*Proof:*</u> No. Assuming that there exists such a polyhedron, we notice that:

$\sum_{f \in F} \deg(f) = 2|E|$. However, the LHS summation contains an odd number of odd integers and is hence odd, while the RHS is even. This is a contradiction.

Example 3.5. *Prove that in every polyhedron there is at least one pair of faces with the same number of sides.*

<u>*Proof:*</u> Let N be the maximum number of sides a face has in the given polyhedron. Then each face has between three and N sides. Notice that the polyhedron must have at least N+1 faces, since two faces can share at most one side and each side of the face with N sides is connected to one other face. Therefore, by the Pigeonhole Principle there must exist a pair of faces with the same number of sides.

Example 3.6. *A polyhedron with n faces is such that there do not exist three faces with the same number of sides. For which n does such a polyhedron exist?*

<u>*Proof:*</u> Assume that we have such a polyhedron. Then there are at most two faces with any number of sides:

$2|E| = \sum_{f \in F} \deg(f) \geq 3 + 3 + 4 + 4 + 5 + 5 + \ldots + (\lceil \frac{n}{2} \rceil + 2) =$

$= 3 + 4 + 5 + \ldots + (\lceil \frac{n}{2} \rceil + 2) + 3 + 4 + 5 + \ldots + (\lfloor \frac{n}{2} \rfloor + 2) =$

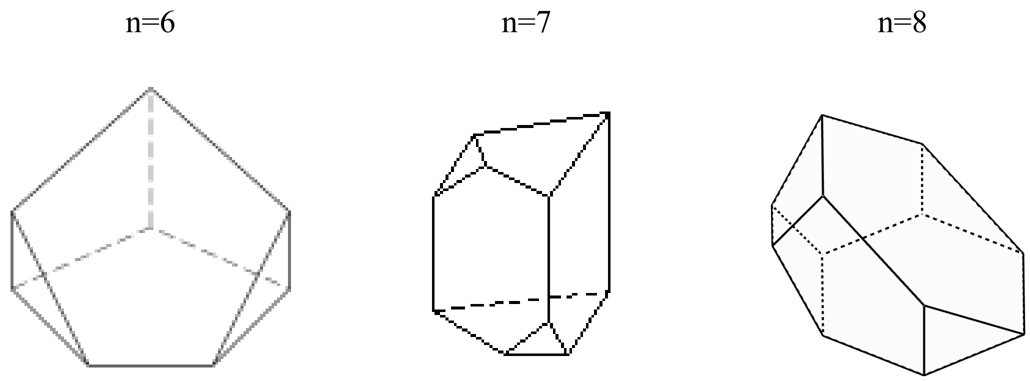
$= \frac{(\lceil \frac{n}{2} \rceil + 2)(\lceil \frac{n}{2} \rceil + 3)}{2} - 3 + \frac{(\lfloor \frac{n}{2} \rfloor + 2)(\lfloor \frac{n}{2} \rfloor + 3)}{2} - 3 = \frac{\lceil \frac{n}{2} \rceil^2 + \lfloor \frac{n}{2} \rfloor^2 + 5\lceil \frac{n}{2} \rceil + 5\lfloor \frac{n}{2} \rfloor}{2} \geq \frac{n^2 + 10n}{4}$. Hence:

$|E| \geq \frac{n^2 + 10n}{8}$ by considering the far LHS and the far RHS. However, notice that

$|E| \leq 3n - 6$. Hence $3|F| - 6 \geq |E| \geq \frac{n^2 + 10n}{8}$. Rearranging we get the inequality:

$1 \geq (n - 7)^2$. Hence we notice that it may only be possible to have a polyhedron without three faces with the same number of sides if $n \in \{6, 7, 8\}$.

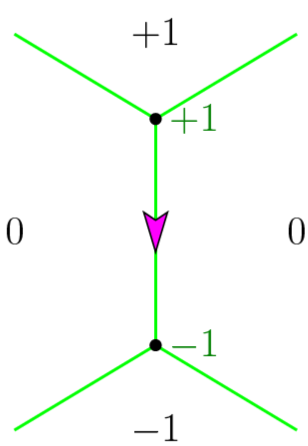
For each such n there exists a polyhedron satisfying the constraints. For completeness, here is one example for each possible number n of faces for such polyhedra:

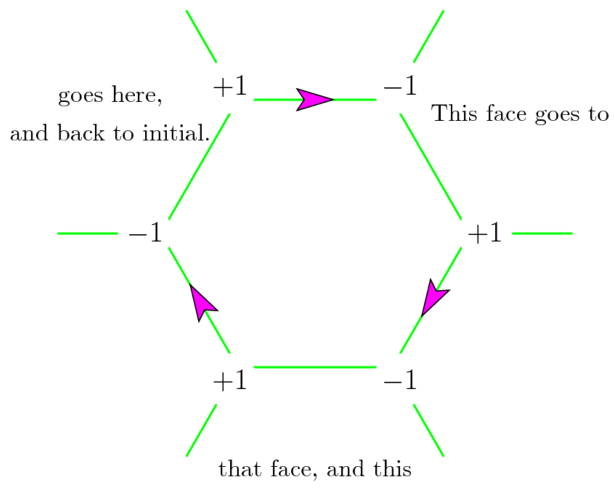


Example 3.7. *Let P be a polyhedron with an odd number of faces, and such that exactly three edges meet at every vertex. Prove that it is possible to assign an integer to each vertex, such that:*

I. *At least one of the integers assigned to the vertices is non-zero.*
II. *For every polygonal face, the sum of the numbers assigned to the vertices of that face equals zero.*

Proof: First we will show that P has a face with an even degree. Assume otherwise, then: $\sum_{f \in F} \deg(f) = 2|E|$. However, we notice that the LHS is odd because it is a sum of an odd number (|F|) of odd integers (deg(f)). Therefore we have reached a contradiction, and hence P has a face with an even degree, let's call it q.
On a different note, we consider a *move* as in the left diagram below:

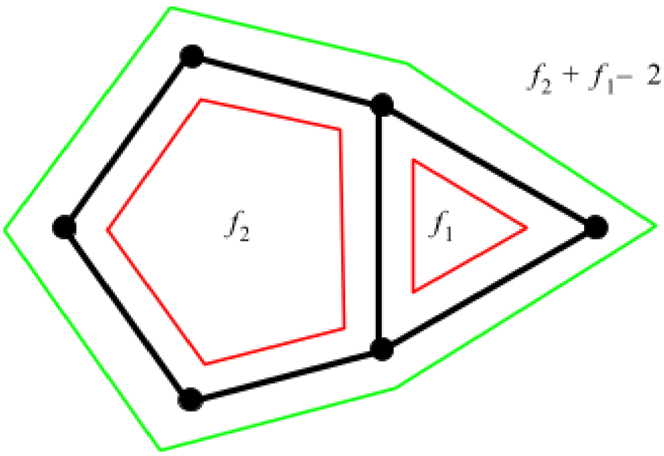




This *move* works by choosing an edge and a direction and changing the number at the start of the edge by +1 and the number at the end of the edge by –1. We notice that a *move* only affects the sum of two faces, changing them by +1 and –1 respectively.
We will now show that the required construction is possible by constructing it ourselves. We begin by setting all numbers at the vertices of the polyhedron to 0. Then we apply $\frac{\deg(q)}{2}$ moves around face q with an even number of sides as in the right diagram above. Next we notice that all affected faces around q have exactly one vertex with the number +1 and one vertex with the number –1, and therefore that the sum of all numbers around each face is exactly 0, as was required.

Example 3.8. *Prove that there exists a cycle along the vertices and edges of any polyhedron of even length.*

<u>*Proof:*</u> If the polyhedron has a face with an even number of sides, this face makes an even cycle. Otherwise all the faces of the polyhedron have an odd number of sides. Choose two such adjacent polygonal faces, with f_1 and f_2 number of sides. Then consider the cycle that goes through all edges of both of these faces, except the edge they share:



This cycle will have an even length of $f_1 + f_2 - 2$. We notice that no edge or vertex appears more than once in this cycle, because our two polygonal sides can only share one edge and two vertices which clearly appear no more than once each in the even cycle.

Example 3.9. *Do all polyhedra have a cycle of odd length?*

Proof: No. Consider the cube. Coloring the vertices in black and white we notice that every edge connects one black and one white vertex. Assume that we have an odd cycle. Choose a direction and a black starting vertex. From this vertex, traverse the polyhedron along this cycle. For each traversed edge, the color of the current vertex flips. Note that the color flips an odd number of times, so when we come back to the starting vertex we will be at a white vertex. However, this is a contradiction.

4. Generalisation

As stated earlier, Euler's formula for polyhedra can be generalized to all orientable[3] surfaces.

4.1 Genus

A simple closed curve is a shape closed by line-segments or/and curved lines. The genus of a surface is an integer that is the largest number of non-intersecting closed simple curves one can draw on the surface without separating it. In other words, it is equal to the number of "holes" on the surface. For example, a sphere has genus zero (no holes), while a torus has genus one (one hole).

Statement 4.1. *Denote the genus of an orientable surface by g. The generalized formula is stated as follows:*

$$|V| - |E| + |F| = \chi(g),$$

where $\chi(g) = 2 - 2g$.

Remark. The proof requires knowledge that is beyond the scope of this paper.

Example 4.1. *Prove that it is impossible for all faces of a polyhedron (not necessarily convex) to be hexagons.*

[3] Orientability. *Wikipedia* https://en.wikipedia.org/wiki/Orientability#Examples

Proof: Notice that $2|E| = \sum_{f \in F} \deg(f) = 6|F| \Rightarrow |E| = 3|F|$, since each hexagon has six edges, and each edge has two hexagons. Now by using $|V| - |E| + |F| = 2$ we have: $|V| = 2|F| + 2$. But since each vertex has to have at least three hexagons, and each hexagon has exactly six vertices: $3|V| \leq 6|F| \Rightarrow |V| \leq 2|F|$. We have: $2|F| + 2 = |V| \leq 2|F|$, which is a contradiction. Hence it is impossible.

5. Conclusion

This text had focused on the applications of Euler's formula. Some applications in both graph theory and regarding polyhedra have been shown. While the formula can prove to be a key tool in some solutions its strength lies in its specificity and as such its strength is also its weakness as it narrows the range of problems that the formula can be applied upon. We also further discussed the generalizations that are possible and these can be further investigated by all interested readers.

6. Additional problems

Problem 7.1. *In a certain small country there are villages, expressways, and fields. Expressways only lead from one village to another and do not cross one another, and it is possible to travel from any village to any other village along the expressways. Each field is completely enclosed by expressways and villages. If there are ten villages and sixteen expressways, then how many fields are there in this country?*

Problem 7.2. *A soccer ball is made up of pentagons and hexagons in such a way that three polygons meet at each vertex. How many pentagons must there be? Prove that no other number of pentagons is possible.*

Problem 7.3. *A certain polyhedron is built entirely from triangles, in such a way that five faces meet at each vertex. Prove that it has to have 20 faces.*

Problem 7.4. *Prove that any planar graph must have a vertex of degree five or less.*

Problem 7.5. *Euler's formula holds for all connected planar graphs. What if a graph is not connected? Suppose a planar graph has two components. What is the value of* $|V| - |E| + |F|$ *now? What if it has k components?*

Problem 7.6. *If you take n points on a circle, then connect every pair of them with a line such that no three lines are concurrent, how many regions do these lines cut the circle into?*

Problem 7.7. *Prove example 3.7 without the constraint that the polyhedron P has an odd number of faces and instead with the additional constraint that P has at least five vertices. Prove that such a construction as in example 3.7 with the tweaked constraints is impossible if P has only four vertices.*