Functional Equations

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Contents

-1 Conventions in this document

In this document, $\mathbb{N}$ *does* include 0. [1]

0 What is a function?

A lot of people go under the assumption that a function is the same thing as a "formula", or rather a closed expression. This is partly true for some of functions, but there are a lot of functions which this does not hold, and can create a lot of confusion.

What a function f really is, is a *relation* over two sets $\mathbb{A}$ and $\mathbb{B}$, which *associates* every element x from the set $\mathbb{A}$ to one element y from $\mathbb{B}$. This association is notated as $f(x) = y$.

A simple, explicit example of a function would be a relation over the sets $\mathbb{A} = \{1, 2\}$, $\mathbb{B} = \{3, 7, 533\}$, which associates $x = 1$ to $y = 3$, $x = 2$ to $y = 7$. We can note that not all values in $\mathbb{B}$ have an x which associates to it. This will be elaborated on in section 3.

This means that a function doesn't have to have a "pattern" or formula to it, or have any kind of logic to it at all, but can be completely arbitrary to what values it gives at different x. Often the functions we deal with do have a pattern or formula, say $f(x) = x^2$. But it is important to remember that this does not have to be the case.

There is a bit of commonly used terminology and notation involving functions. The set of x-values, above reffered to as $\mathbb{A}$ is called the *domain* of the function, and the set of y-values, $\mathbb{B}$ above, is called the *codomain* of f. To consicely describe that a function has a domain $\mathbb{A}$ and codomain $\mathbb{B}$, we use the following notation: $f : \mathbb{A} \to \mathbb{B}$

1 Introduction to functional equations

Functional equations are a common type of problem where an unknown function is to be found, given that the function satisfies a given equality.

The functional equation problems given in competitions usually consist of two pieces of information:

1. The unknown function's domain and codomain.
2. The equality/equalities fulfilled by the function.

The functional equation is usually quantified over some of the function's variables. For example, it might be given that the equality $f(x) = xf(x+1)+1$

[1] This varies a lot from country to country. If you compete internationally, make sure to note which $\mathbb{N}$ is used.

should hold for any integer value of x. You could think of it like an (sometimes uncountably) infinite system of equations, one for each value of x.

$$\begin{aligned} &\vdots \\ f(-1) &= -1 \cdot f(0) + 1 \\ f(0) &= 0 \cdot f(1) + 1 \\ f(1) &= 1 \cdot f(2) + 1 \\ f(2) &= 2 \cdot f(3) + 1 \\ &\vdots \end{aligned}$$

Of course, if something is true in general, it is also true in particular. Usually functional equation solving is about limiting yourself to some interesting subset of x (or whatever your variable(s) might be) and inferring useful information about the function through the set of equations generated. For the example above, we can see that when $x = 0$, we get the equality $f(0) = 1$ which provides us a potentially useful point of entrance to the problem.

A *very important* thing to take into consideration when dealing with functional equations in math competitions is to *verify your answer* by substituting it back into the equation. In general, when finding all solutions to an equation (or system of equations), unless all steps in your proof are equivalences, you "silently" assume that there exists a solution for the equation(s). However this might not always be the case. You need to make sure that after you have proven:

There exists a value/function f satisfying the equation
$\Downarrow$
f is one of the values/functions I found

that you examine the case:

There doesn't exist a value/function f satisfying the equation

This is done simply by checking if the solutions you found by *assuming* that the equation had a solution are correct or not. This is often forgotten in all the excitement of solving a problem and hurry to move on to the next one, and sadly costs many people important points whenever functitonal equations in particular come up.

2 Substitutions and systems of equations

As previously mentioned, a common approach to solving functional equations is substitution. As the equation is quantified over all x we can substitute $x \to z$ for any z. By making a good choice of z one can either solve the equation directly or reach a solvable system of equations. Note that the different equations in the system don't need to share the same substitution.

An example of a functional equation solvable with this technique is problem 2 from the SMT final 2018:

$$\text{Find all } f : \mathbb{R} \to \mathbb{R} \text{ such that:}$$
$$f(x) + 2f\left(\sqrt[3]{1-x^3}\right) = x^3$$

We can do the substitution $x \to \sqrt[3]{1-x^3}$, and end up with:

$$f\left(\sqrt[3]{1-x^3}\right) + 2f\left(\sqrt[3]{1-(\sqrt[3]{1-x^3})^3}\right) = \left(\sqrt[3]{1-x^3}\right)^3$$
$$f\left(\sqrt[3]{1-x^3}\right) + 2f(x) = 1 - x^3$$

We can combine this new equation with the original in a system of equations:

$$\begin{aligned}
&\begin{cases} f(x) + 2f\left(\sqrt[3]{1-x^3}\right) &= x^3 \\ 2f(x) + f\left(\sqrt[3]{1-x^3}\right) &= 1 - x^3 \end{cases} \implies \\
\implies\quad &\begin{cases} f(x) + 2f\left(\sqrt[3]{1-x^3}\right) &= x^3 \\ 4f(x) + 2f\left(\sqrt[3]{1-x^3}\right) &= 2 - 2x^3 \end{cases} \implies \\
\implies\quad & 4f(x) - f(x) + f\left(\sqrt[3]{1-x^3}\right) - 2f\left(\sqrt[3]{1-x^3}\right) = 2 - 2x^3 - x^3 \implies \\
\implies\quad & 3f(x) = 2 - 3x^3 \implies \\
\implies\quad & f(x) = \frac{2}{3} - x^3
\end{aligned}$$

Remember to verify that this is a solution!

$$\begin{aligned}
x^3 &= f(x) + 2f\left(\sqrt[3]{1-x^3}\right) \iff \\
\iff x^3 &= \frac{2}{3} - x^3 + 2\left(\frac{2}{3} - \sqrt[3]{1-x^3}^3\right) \iff \\
\iff x^3 &= \frac{2}{3} - x^3 + \frac{4}{3} - 2 + 2x^3 \iff \\
\iff x^3 &= x^3 \quad \blacksquare
\end{aligned}$$

Thus, we have the solution $f(x) = \frac{2}{3} - x^3$

Also previously mentioned is another substitution technique, reducing the values of x to some subset of the domain. Most often, only one value of x is considered at a time. This can often give information about certain points of the function. Common values to try are 0, 1 and -1. When dealing with equations containing more than one independant variable, substituting $x = y$ is also a common approach. What other values to try often depends on the problem. As a general rule of thumb, if a substitution cancels something, it's worth trying.

An example of a problem solvable using this type of substitution is the following:

$$\text{Find all } f : \mathbb{R} \to \mathbb{R}$$
$$f(xy) = 3f(x + y^2 - 1) + 4(x + y)^2$$

Substituting $y = 1$ reduces the problem into:

$$\begin{aligned} f(x) &= 3f(x + 1 - 1) + 4(x + 1)^2 \\ -2f(x) &= 4(x + 1)^2 \\ f(x) &= -2(x + 1)^2 \end{aligned}$$

Remember to verify!

$$\begin{aligned} f(x) &= 3f(x + y^2 - 1) + 4(x + y)^2 \\ -2(x + 1)^2 &= 3(-2(x + y^2)^2) + 4(x + y)^2 \end{aligned}$$

This is not always true! For example, when $x = y = 0$, the equality clearly doesn't hold. Thus, our proposed solution is not a true solution to the equation, which means that f does not exist. This is why it is always important to check!

Often though, a substitution setting a variable to a single value, like this, only gives us information about a certain value of f, which can later be used when using other techniques. An example of this is the following problem, adapted from a problem at the 2020 Lund Math Camp, where setting y to a certain value gives us information required to solve the general case:

$$\text{Find all } f : \mathbb{R} \to \mathbb{R} \text{ such that:}$$
$$f(x^2 + y) = f(x^{27} + 2y) + f(x^4)$$

We can intuitively see that $x^2 + y$ and $x^{27} + 2y$ are quite "complicated" and difficult to manipulate. It would be nice if we could make them equal so that they cancel. This is done by letting $y = x^2 - x^{27}$:

$$\begin{aligned} f(x^2 + (x^2 - x^{27})) &= f(x^{27} + 2(x^2 - x^{27})) + f(x^4) \\ f(2x^2 - x^{27}) &= f(2x^2 - x^{27}) + f(x^4) \\ 0 &= f(x^4) && b = x^4 \geq 0 \\ 0 &= f(b) \end{aligned}$$

Thus we have proved that all non-negative values of b gives $f(b) = 0$ Let us substitute $y = 0$ into the original equation:

$$\begin{aligned} f(x^2 + y) &= f(x^{27} + 2y) + f(x^4) \\ f(x^2) &= f(x^{27}) + f(x^4) \qquad (x^2 \geq 0, x^4 \geq 0) \\ 0 &= f(x^{27}) + 0 \end{aligned}$$

Since x^{27} can attain any value, we find that f is constant 0. We can easily verify this solution:

$$\begin{aligned} f(x^2 + y) &= f(x^{27} + 2y) + f(x^4) \\ 0 &= 0 + 0 \quad \blacksquare \end{aligned}$$

3 Injectivity, surjectivity and bijectivity

Three important properties among functions are injectivity, surjectivity and bijectivity. Bijectivity is equivalent to a function being both injetive and surjective, but gives us a lot of nice properties to work with.

We need to introduce one piece of terminology when dealing with $\cdots$-jectivities: the *image* of a function. As we saw in section 0, a function does not have to associate something to every value in the codomain. We define the *image* to be every value in the codomain which is associated by some value in the domain. For the example in section 0, $f : \{1, 2\} \to \{3, 7, 533\}$, $f(1) = 3, f(2) = 7$, the value 533 is the only value with no x associated with it. Therefore, the image of f is $\{3, 7\}$.

Injectivity means that for every element y in the codomain, there exists *at most one* element x in the domain such that $f(x) = y$.

Surjectivity means that for every element y in the codomain there exists *at least one* element x in the domain such that $f(x) = y$. Surjectivity is equivalent to the image being equal to the codomain.

Bijectivity, as previously stated, is the conjunction of injectivity and surjectivity: for every element y in the codomain there exists *one and only one* element x in the domain such that $f(x) = y$. Bijectivity gives us the nice property that a function has an unique inverse. The inverse of a function $f : \mathbb{A} \to \mathbb{B}$ is notated $f^{-1} : \mathbb{B} \to \mathbb{A}$ and defined by $f(f^{-1}(y)) = y$ and $f^{-1}(f(x)) = x$. One can see that both injectivity and surjectivity is needed for the inverse to be uniquely defined: if a function lacks injectivity, the inverse will be defined to be multiple values at a single point, and if a function lacks surjectivity, there will be some points in the domain of the inverse which are not defined

Some examples of functions and their $\cdots$-jectivity are:

1. $f : \mathbb{R} \to \mathbb{R}, f(x) = 7$
   Neither injective (there are more than one x such that $f(x) = 7$) nor surjective (all values in $\mathbb{R}$ except 7 lack a corresponding x).

2. $f : \mathbb{R} \to \mathbb{R}, f(x) = \tan x$
   Not injective ($f(x) = y \Rightarrow f(x + \pi) = y$), but surjective (since the image of tan is $\mathbb{R}$).

3. $f : \mathbb{R} \to \mathbb{R}, f(x) = 7 + 3x$
   Injective (can be proven easily by contradiction) and surjective (also easily verified), which means f is bijective. We can thus find the inverse of f: $f^{-1}(y) = \frac{y-7}{3}$

4. $f : \mathbb{Q} \setminus \{0\} \to \mathbb{Q}, f(x) = x^{-1}$
   Injective ($x^{-1} = y^{-1} \Rightarrow x = y$), but not surjective (there is no x such that $f(x) = 0$).

5. $f : \mathbb{Q} \setminus \{0\} \to \mathbb{Q} \setminus \{0\}, f(x) = x^{-1}$
   Injective (Same as for 4.), and surjective (can easily be verified), thus f is bijective. We can find the inverse of f: $f^{-1}(y) = y^{-1}$ (Coincidentally the same as f. Excercise: find other (families of) function which are their own inverse. These are called *involutions.*).

Note that 4. and 5. both share the same "equality", $f(x) = x^{-1}$, but have different $\cdots$-jectivities. This is surprising to many, but is easy to understand once you realize that the domain and codomain are just as part of the definition of a function as the "equality" is. Therefore, one must be sure to take the domain and codomain into account when dealing with functional equations.

Proving surjectivity is often as simple as having $f(...)$ =(something that can take all values). Note that it doesn't matter if there is a very complicated expression as argument to f. After surjectivity is proven, you are free to set $x = x_v$ such that $f(x_v) = v$ for any v.

Example problem from SMT-final 2014:

$$f : \mathbb{R} \to \mathbb{R}$$

$$f(f(x + y) - f(x - y)) = xy$$

Solution: Note that we can easily get any real number in the right hand side, and therefore also the left hand side and attain any value, which means that f is surjective.

Performing the substitution $y \to -y$ gives:

$$\begin{aligned} f(f(x-y) - f(x+y)) = -xy &\iff \\ f(-(f(x+y) - f(x-y))) = -xy &= \\ = -f(f(x+y) - f(x-y)) &\iff \\ f(-a) = -f(a) \qquad & a = f(x+y) - f(x-y) \end{aligned}$$

By setting $x = y$, we have $a = f(2x) - f(0) \iff a + f(0) = f(2x)$. As f is surjective, the right hand side can take all values, which means that a can do the same. Thus, we have $f(-a) = -f(a)$ for all a. Lastly, let $x = 0$:

$$f(f(y) - f(-y)) = 0 \iff f(2f(y)) = 0$$

which by the surjectivity implies that $f(x) = 0$ for all x.

Remember to check the answer! $f(x) = 0$ would imply f is not surjective, and is therefore not a solution to the problem. We can therefore conclude that no such functions exist.

4 Functional equations over the integers

Functional equations over $\mathbb{N}$ (and $\mathbb{Z}$) are often solved using induction. Induction is a general technique for mathematical proofs of statements quantified over the natural numbers. To prove a statement $P(x), x \in \mathbb{N}$, it is sufficient to (1) prove that $P(0)$ is true, and (2) that $P(n-1) \implies P(n)$ for $n > 0$, this is what induction does. If we have these proofs, we can intuitively see that the statement holds for any number m by "chaining" the seconds step that many times:

$$\begin{aligned}
P(0) &\text{ is true} && \text{from (1)}\\
P(0) &\implies P(1) && \text{from (2)}\\
P(1) &\implies P(2) && \text{from (2)}\\
&\dots\\
P(m-2) &\implies P(m-1) && \text{from (2)}\\
P(m-1) &\implies P(m) && \text{from (2)}\\
\text{Hence, } P(m) &\text{ is true.} && \text{QED.}
\end{aligned}$$

We can use following problem to demonstrate using induction to solve functional equations over whole numbers:

$$f : \mathbb{N} \to \mathbb{N} \tag{1}$$

$$f(x) = f(x-1) + 1 \tag{2}$$

Let $c = f(0)$. We can see that $f(1) = f(0)+1 = c+1$, $f(2) = f(1)+1 = c+2$. We would like to prove that $f(x) = x + c$, using induction. Our statement we would like to prove is thus $P(x) \iff f(x) = x + c$.

To do our proof by induction, we start by proving the base case, $P(0) \iff f(0) = 0 + c = c$. However, we defined c to be exactly $f(0)$, so this comes by definition.

Next, we need to prove $P(n-1) \implies P(n)$.

$$
\begin{aligned}
P(n-1) &\iff f(x-1) = x-1+c \iff \\
&\iff \overbrace{f(x) = f(x-1)+1}^{\text{Stated in the problem}} = (x-1+c)+1 = x+c \iff \\
&\iff P(n)
\end{aligned}
$$

Thus, we have proved $P(0)$ and $P(n-1) \implies P(n)$ for $n > 0$, so $P(n)$ holds for all n, and we have proven our proposition.

This is one of the rare cases where every step in the proof was an equivalence. This means that we don't need to check that our solution always holds, however you should still do this for good measure.

If a functional equation has a domain of the whole numbers $\mathbb{Z}$, we'll have to change the approach a little bit. Regular induction only works for proving a statement for all the natural numbers, $P(n-1) \implies P(n)$ doesn't work "in reverse". To use induction over $\mathbb{Z}$, there are a few (mostly equivalent) approaches to take:

- One way is to prove the statement for the negative numbers separately. One can start by proving $P(n)$ for $n \geq 0$ using regular induction, then construct a $Q(n) \iff P(-n)$ and prove $Q(n)$ for $n \geq 0$ using induction.
- One can also equivalently prove the negative case by showing $P(n+1) \Rightarrow P(n)$ for $n < 0$
- If your proof that $P(n) \Rightarrow P(n+1)$ is an equivalence, that is you've proved $P(n-1) \iff P(n)$, and your proof does not rely on n being greater than 0, then you have proved $P(n)$ for all $n \in \mathbb{Z}$. This approach might complicate the second part of the proof by induction a bit, but saves work as you don't need to prove anything else.

We can note that in our solution for the problem above, the inductive step is an equivalence and does not rely on n being greater than one. Thus, the exact same solution works for $f : \mathbb{Z} \to \mathbb{Z}$ as well.

Sometimes induction will only give you the value of $f(x)$ for x in a subset of Z or N. Then you might need to find a clever way of filling in the gaps. Try this out by attempting the following problem:

Find all f such that:

$$
\begin{aligned}
&f : \mathbb{Z}^+ \to \mathbb{Z}^+ \\
&f(n+1) > f(n), f(n^3) = f(n)^3
\end{aligned}
$$

5 Classical equations

There are a few well known functional equation whose solutions might be useful to be aware of. By rewriting and substituting, one can some times reduce a given functional equation to one of these. Listed in Table 1 are some classical equations together with their solutions.

$f(x+y)=f(x)+f(y)$	$f(x)=cx$
$f(x+y)=f(x)f(y)$	$f(x)=c^x$
$f(xy)=f(x)+f(y)$	$f(x)=c\cdot\ln(x)$
$f(xy)=f(x)f(y)$	$f(x)=x^c$

Table 1: Some classical functional equations and their solutions

However, for these solutions to be unique, some additional condition is required. The function being continious or the function being monotonous are both conditions that imply that the solution is of the form above.

It should be noted that all of these equations are in fact the same. By some simple substitutions and rearranging, they can all be brought to the same form. Can you spot how? Hint: Try $x = ln(a), y = ln(b)$

5.1 The Cauchy Equation

The first equation in the list above is often refered to as the Cauchy Equation:

$$f(x+y) = f(x) + f(y) \tag{3}$$

We will study the solution of this equation, as it utilizes technique that can be useful for other functional equations aswell.

In the most general case we are looking for functions $f : \mathbb{R} \to \mathbb{R}$ such that all $x, y \in \mathbb{R}$ satisfy (3). However, to solve this we will start by only considering the integers.

By letting $x = 0$ och $y = 0$ we get imediatly get that $f(0) = 0$. Define $c = f(1)$. By letting $(x, y) = (1, 1)$, we get $f(2) = f(1) + f(1) = 2c$. By letting $(x, y) = (2, 1)$, we get $f(3) = 2c + f(1) = 3c$. Continuing like this, we get by induction that $f(N) = cN$ for all positive integers N. We can substitute x for $x - y$ in (3) and get:

$$f(x-y) = f(x) - f(y) \tag{4}$$

Doing a similar induction for this form gives us that $f(N) = cN$ for all integers N.

We managed to solve the cauchy equation for the integers. But this can easily be generalised. If we instead of taking $c = f(1)$ take $c' = f(\alpha)$ for some $\alpha \in \mathbb{R}$, and considering $(x, y) = (\alpha N, \alpha)$ instead of $(x, y) = (N, 1)$, we get:

$$f(\alpha N) = N f(\alpha) \tag{5}$$

By taking $\alpha = \frac{1}{q}$, we have $f(\frac{N}{q}) = N f(\frac{1}{q})$. But taking $N = q$ now implies

$$qf(\frac{1}{q}) = f(\frac{q}{q}) = f(1) = c \Longrightarrow f(\frac{1}{q}) = \frac{c}{q} \Longrightarrow f(\frac{p}{q}) = c\frac{p}{q}$$

Now the last hurdle to cross would be to go from the rational numbers to the real numbers. This is when the extra condition is needed. The Cauchy Equation can actually be solved with an even weaker condition than continuity or monoticity. It is enough that there exist some interval where f is bounded. That is, there exists constants $a, b, c \in \mathbb{R}$ such that $|f(x)| \leq c$ for all $x \in [a, b]$. See appendix for proof of this.

Example problem:

$$f : \mathbb{R} \to \mathbb{R}, f \text{ is continuous}$$
$$f(x + f(y)) = f(x) - y$$

Solution: If we fix x, it is clear that we can get the right hand side to any value we want by varying y. f must therefore be surjective. Now we can take $y = y_0$ so that $f(y_0) = 0$:

$$f(x + f(y_0)) = f(x) - y_0 \Longrightarrow f(x) = f(x) - y_0 \Longrightarrow y_0 = 0 \Longrightarrow f(0) = 0$$

In a sense, we have now extracted what we can from letting $y = 0$, so to it is reasonable to instead try $x = 0$:

$$f(0 + f(y)) = f(0) - y \Longrightarrow f(f(y)) = -y$$

To be able to use this new identity we have, we would like to make $f(f(y))$ appear somewhere. To achieve this, we can simply replace y with $f(y)$:

$$f(x+f(f(y))) = f(x)-f(y) \Longrightarrow f(x-y) = f(x)-f(y) \Longrightarrow f(x) = f(y)+f(x-y)$$

By finally replacing x with $x + y$ we get the Cauchy Equation:

$$f(x + y) = f(x) + f(y)$$

and as f is continuous the solutions are $f(x) = cx$. Remains to plug into the original equation to see what values of c are valid.

6 More practice problems

- SMT-kval 2007: Find all $f : \mathbb{R} \to \mathbb{R}$ such that

$$x(f(x) + f(-x) + 2) + 2f(-x) = 0 \text{ for all } x \in \mathbb{R}$$

- Lundläger 2020: Find all $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(f(x)^2 + f(y)) = xf(x) + y \text{ for all } x, y \in \mathbb{R}$$

- Lundläger 2020: Find all $f : \mathbb{R} \to \mathbb{R}$ such that

$$(x-y)f(x+y) - (x+y)f(x-y) = 4xyf(x^2-y^2) \text{ for all } x, y \in \mathbb{R}$$

- Baltic Way 2017: Find all $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x^2y) = f(xy) + yf(f(x)+y) \text{ for all } x \in \mathbb{R}$$

- Baltic Way 1995: Find all $f : \mathbb{Z} \to \mathbb{Z}$ such that

$$f(x) = f(x^2+x+1) \text{ for all } x \in \mathbb{Z}$$

where a) f is even ($f(-x) = f(x)$ for all x)
b) f is odd ($f(-x) = -f(x)$ for all x)

- SMT-final 2019: Find $f(2019)$ where $f : \mathbb{Z}^+ \to \mathbb{Z}^+$ is such that

$$f(n+1) > f(n)$$

$$f(f(n)) = 3n$$

- IMO 2019: Find all $f : \mathbb{Z} \to \mathbb{Z}$ such that

$$f(2a) + 2f(b) = f(f(a+b)) \text{ for all } a, b \in \mathbb{Z}$$

A Pathological solutions to the Cauchy Equation

A solution to some problem is often said to be "pathological" if it satisfies the given constraints, but behave in difficult to handle and counterintuitive ways. The opposite of pathological would be "well behaved". We shall prove that in the context of the Cauchy Equation, the only well behaved solutions are $f(x) = cx$. More specifically, we shall prove that for all other solutions, the graph $y = f(x)$ is dense in the plane. This means that for every point P in the plane, there is a point on the graph $y = f(x)$ arbitrarily close to P. This property clearly cannot be satisfied if the function is continuous, monotonious or bounded on some interval.

Define $g(x) = f(x\alpha)$ for some real number α. If f satisfies the Cauchy Eqaution, so does g. By our proof above:

$$g(\frac{p}{q}) = \frac{p}{q}g(1) \iff f(\frac{p}{q}\alpha) = \frac{p}{q}f(\alpha) \tag{6}$$

Now note that if (x_1, y_1) and (x_2, y_2) are points on the graph, then $(x_1 + x_2, y_1+y_2)$ is aswell. This is because $(x_1+x_2, y_1+y_2) = (x_1+x_2, f(x_1)+f(x_2)) =$

$(x_1+x_2, f(x_1+x_2))$. Also, if (x_1, y_1) is a point in the graph, then $(\frac{p}{q}x_1, \frac{p}{q}y_1)$ is aswell, as $(\frac{p}{q}x_1, \frac{p}{q}y_1) = (\frac{p}{q}x_1, \frac{p}{q}f(x_1)) = (\frac{p}{q}x_1, f(\frac{p}{q}x_1))$ (by (6)). Combining this gives us that any linear combination with rational coefficients of two points on the graph is on the graph.

Now if $f(x) = cx$ doesn't hold for all x, then there is some point $y_1 = f(x_1)$ where $y_1 \neq cx_1$. Therefore the points (x_1, y_1) and $(1, c)$ are linearly independent, which means that every point on the plane can be expressed as a linear combination of them, albiet with real coefficients. However, there are rational numbers arbitrarily close to any real numbers, which means that a rational linear combination of the two points can get arbitrarily close to any point on the plane.

Now, a natural question is whether or not these pathological solutions exist at all? It turns out that they indeed do, and the proof relies on the fact that all vector spaces have a basis. The details is not covered here. Note that the fact that the pathological solutions can be called "ill behaved" shouldn't undermine their mathematical significance. In fact, the existence of these solutions helped solve Hilberts 3rd problem, as an example.