Mathematical induction in different contexts

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1 Introduction

In this paper we will discuss the *principle of mathematical induction* and its use for proofs in different mathematical fields. We will explore problems in algebra such as identities, inequalities, functional equations over $\mathbb{N}$ and after that we will look at some important theorems in number theory and graph theory. In the end we will investigate a rather interesting field, combinatorial geometry, which combines the objects from geometry and the ideas from combinatorics. You will quickly notice that mathematical induction is a powerful technique which you definitely should add to your tool box.

2 What is mathematical induction?

Let $P(n)$ be a statement involving the variable n such that $P(n)$ is either true or false. Suppose that

- $P(k_0)$ is true.
- If $P(k)$ is true for some integer $k \geq k_0$, then $P(k+1)$ is true.

Then $P(n)$ is true for all integers $n \geq k_0$. To demonstrate what this actually means, let us suppose that the statement $P(0)$ is true and, if $P(k)$ is true for some integer $k \geq 0$, then $P(k+1)$ is true. We know that $P(0)$ is true and this implies that $P(1)$ is true. But if $P(1)$ is true then $P(2)$ must also be true. But if $P(2)$ is true then $P(3)$ must also be true, and so on. Domino bricks, where the fall of one brick causes the fall of the next one and so on, is a good and often used parable for this. As you can see we only have to prove two things and usually we give these two steps a name. The first step is called *the base case*, which is that we need to show $P(k_0)$ is true for the integer k_0. The next step is called *the inductive step*, notice that in this step we are assuming that $P(k)$ is true for some $k \geq k_0$ and this is called *the induction hypothesis*.

With this technique you will get far, but sometimes we need to prove something stronger or split up *the inductive step*. The first variation, *strong induction*, is similar to the one previously discussed.

Let $P(n)$ be a statement involving the variable n such that $P(n)$ is either true or false. Suppose that

- $P(k_0)$ is true.
- If $P(k_0), P(k_0+1), \ldots, P(k)$ are true for some integer $k \geq k_0$, then $P(k+1)$ is true.

Then $P(n)$ is true for all integers $n \geq k_0$. Try to convince yourself why this is true.

In the previous two variations there must be a connection between $P(k)$ and $P(k+1)$. But what if this is not the case? Perhaps we can find a connection between $P(k)$ and $P(k+2)$. How would we proceed then? In this case we must have two *base cases*.

Let $P(n)$ be a statement involving the variable n such that $P(n)$ is either true or false. Suppose that

- $P(k_0)$ and $P(k_0+1)$ are true.
- If $P(k)$ is true for some integer $k \geq k_0$, then $P(k+2)$ is true.

Then $P(n)$ is true for all integers $n \geq k_0$. Try to convince yourself why this is true. We can easily generalize this to a connection between $P(n)$ and $P(n+k)$. How many *base cases* do we need now? There are a lot of different variations we could talk about but this goes beyond the scope of this paper. If you want to see some other variations we recommend *forward-backward induction* which is used to prove the AM-GM inequality. The AM-GM inequality states that for nonnegative real numbers $x_1, x_2 \ldots, x_n$, the following holds

$$\frac{x_1 + x_2 + \cdots + x_n}{n} \geq \sqrt[n]{x_1 \cdot x_2 \cdots x_n}.$$

In the case $n = 2$, we have $\frac{x_1+x_2}{2} \geq \sqrt{x_1 x_2}$ which is true since it is equivalent to $(\sqrt{x_1} - \sqrt{x_2})^2 \geq 0$. The idea is to prove that the inequality is true for $n = 2^m$, where m is any positive integer, this acts as the "forward" part. This can be done by proving that if $P(k)$ is true for some integer $k \geq 2$ then $P(2k)$ is true, with the base case $P(2)$. To prove the inequality for all integers $n \geq 2$ we have to go "backwards". This can be done by proving that if $P(k)$ is true for some integer $k \geq 3$ then $P(k-1)$ is true. Since n can be arbitrarily large and that we can go "backwards" the inequality holds for all integers $n \geq 2$. Another useful variation is *multidimensional induction* when we have a statement that involves several variables.

3 Algebra

In this section we will explore problems in algebra that can be solved by induction. Some of the problems that can be solved using this technique are identities, inequalities and functional equations. We will do that but first let us consider the following classical identity.

Example 3.1. Show that for all integers $n \geq 1$,

$$1 + 2 + 3 + \ldots + (n-2) + (n-1) + n = \frac{n(n+1)}{2}.$$

Proof. If $n = 1$ we have $1 = \frac{1(1+1)}{2}$ which clearly holds. Let us assume that the equation in the problem statement holds for the particular value $m \in \mathbb{Z}^+$ (this is the induction hypothesis). Note here that m is a particular value for the variable n. Later the letter n will represent both of these concepts at the same time. That is

$$1 + 2 + 3 + \ldots + (m-2) + (m-1) + m = \frac{m(m+1)}{2}.$$

By adding $m+1$ to both sides and rewriting the right hand side we get

$$\begin{aligned} 1+2+3+\ldots+(m-1)+m+(m+1) &= \frac{m(m+1)}{2}+(m+1) \\ &= \frac{m(m+1)}{2}+\frac{2(m+1)}{2} \\ &= \frac{(m+1)(m+2)}{2}, \end{aligned}$$

which means it also holds for the value $m+1$. By the principle of induction the problem is solved. □

Notice that we did not transform one expression into another but instead showed that two things shared a property of growth. As we increased the variable n by 1 the expressions both grew by $n+1$. Since they both started at 1 they must be equal.

Example 3.2 (Bernoulli's inequality)**.** Show that for all integers $n \geq 1$ and all real numbers $x \geq -1$ the following inequality holds

$$(1+x)^n \geq 1+nx.$$

Proof. If $n=1$, we have $1+x \geq 1+x$ which is true. Assume that

$$(1+x)^k \geq 1+kx$$

for some positive integer k. Notice that

$$\begin{aligned} (1+x)^{k+1} &= (1+x)(1+x)^k \\ &\geq (1+x)(1+kx) \quad \text{(Induction hypothesis and } 1+x \geq 0) \\ &= 1+(k+1)x+kx^2 \\ &\geq 1+(k+1)x. \quad (kx^2 \geq 0) \end{aligned}$$

By the principle of mathematical induction we are done. □

Example 3.3. Show that for all integers $n \geq 1$,

$$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\ldots+\frac{1}{n^2} \leq 2-\frac{1}{n}.$$

Proof. If $n=1$ we have $\frac{1}{1^2} \leq 2-\frac{1}{1}$ which clearly holds. Let us assume that the equation holds for the particular value $n \in \mathbb{Z}^+$. That is

$$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\ldots+\frac{1}{n^2} \leq 2-\frac{1}{n}.$$

By adding $\frac{1}{(n+1)^2}$ to both sides of the inequality we get

$$\frac{1}{1^2}+\frac{1}{2^2}+\ldots+\frac{1}{n^2}+\frac{1}{(n+1)^2} \leq 2-\frac{1}{n}+\frac{1}{(n+1)^2}.$$

But since

$$-\frac{1}{n}+\frac{1}{(n+1)^2} \leq -\frac{1}{n}+\frac{1}{n(n+1)} = -\frac{n+1}{n(n+1)}+\frac{1}{n(n+1)} = -\frac{n}{n(n+1)} = -\frac{1}{n+1},$$

we have

$$\frac{1}{1^2}+\frac{1}{2^2}+\ldots+\frac{1}{n^2}+\frac{1}{(n+1)^2} \leq 2-\frac{1}{n+1}.$$

This means that the inequality holds for the value $n+1$. By the principle of induction, the inequality holds for every value of $n \in \mathbb{Z}^+$. □

Another type of algebra problem that effectively uses induction is functional equations. Consider the following example.

Example 3.4. Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that

$$f(n+1) - f(n) = n+2,$$

for all $n \in \mathbb{N}$.

Proof. Let $k = f(0)$. We will now show that $f(n) = k + \frac{n(n+3)}{2}$. Clearly it holds for $n = 0$. Now assume that it holds for some n. Then we get

$$f(n+1) - (k + \frac{n(n+3)}{2}) = n+2.$$

By rearranging we get the following

$$\begin{aligned} f(n+1) &= k + \frac{n(n+3)}{2} + n + 2 \\ &= k + \frac{n(n+3)+2n+4}{2} \\ &= k + \frac{n^2+5n+4}{2} \\ &= k + \frac{(n+1)(n+4)}{2}. \end{aligned}$$

But this means that our formula holds for $n+1$ as well so by induction it must hold for every nonnegative integer which means that the solutions are precisely those described by $f(n) = k + \frac{n(n+3)}{2}$ for all $n \in \mathbb{N}$, where $k \in \mathbb{N}$. Clearly this function satisfies the given equation. □

You might wonder where we got the expression to begin with. The easiest way to find it is to plug in small numbers such as $0, 1, 2$ and 3 and try to find a pattern within the values of $f(n)$. You would find that $f(0) = k$, $f(1) = k+2$, $f(2) = k+5$, $f(3) = k+9$. As you can see, the numbers 0, 2, 5 and 9 here grow by +2, +3, +4 and so on. This suggests that the formula which describes them should be a quadratic since the difference between them is linear (one degree lower). The quadratic can then be found by setting $f(n) = an^2 + bn + c$ and solving for a, b and c.

3.1 Problems

Problem 3.1. Determine a formula for the sum $S_n = 1\cdot 2+2\cdot 3+\ldots+n(n+1)$.

Problem 3.2. Prove that $n! > 2^n$ for every integer $n \geq 4$.

Problem 3.3. Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that

$$nf(n+1) - (n+2)f(n) = n+6,$$

for all $n \in \mathbb{N}$.

4 Number theory

Since both number theory and induction primarily consider the integers, induction is a valuable tool for solving and proving many of the number theoretic problems and theorems. This section will look at some of these, starting by proving two very important theorems - the fundamental theorem of arithmetic and Fermat's little theorem.

Example 4.1 (Fundamental theorem of arithmetic)**.** Show that every positive integer $n \geq 2$ can be written as a product of prime numbers.

Proof. We proceed by strong induction. The base case $n = 2$ is true because 2 is a prime number. Assume that the statement is true for $n = 2, 3, \ldots, k$ for some integer $k \geq 2$. Now we want to show that $k+1$ can be prime factorized and we can do this by noting that $k+1$ is either a prime number or a composite number. If $k+1$ is a prime number then the statement is true. If $k+1$ is a composite number then we can write $k+1 = ab$, where $a, b \in \mathbb{Z}^+$. Notice that $a, b \leq k$ and we can apply the induction hypothesis and it follows that $ab = k+1$ can be prime factorized. We are done. *Since this paper is dedicated to induction proofs only, we leave proving the uniqueness of the factorization to the reader.* □

Example 4.2 (Fermat's little theorem)**.** If p is a prime number, show that $p \mid a^p - a$, where a is any integer.

Proof. We start with the induction basis. Since $p \mid 1^p - 1 = 0$, the theorem is valid for $a = 1$. Now assume it applies for some integer a, that is,

$$p \mid a^p - a$$

For the induction step we need to prove $p \mid (a+1)^p - (a+1)$. The binomial expansion of $(a+1)^p$ gives,

$$(a+1)^p - (a+1) = \sum_{m=0}^{p} \binom{p}{m} a^m - (a+1) = a^p + 1 + \sum_{m=1}^{p-1} \binom{p}{m} a^m - (a+1)$$

or

$$(a+1)^p - (a+1) = a^p - a + \sum_{m=1}^{p-1} \binom{p}{m} a^m$$

Note that $p \mid \binom{p}{m}$ for $1 \leq m \leq p-1$, because p is a prime number. This can be seen by rewriting as

$$\binom{p}{m} = \frac{p!}{m! \cdot (p-m)!}$$

For $1 \leq m \leq p-1$, p divides the numerator - but not the denominator since p is a prime number. Since $\binom{p}{m} \in \mathbb{Z}$, this means $p \mid \binom{p}{m}$. Our hypothesis says $p \mid a^p - a$, and we have,

$$p \mid a^p - a + \sum_{m=1}^{p-1} \binom{p}{m} a^m = (a+1)^p - (a+1)$$

and the theorem is proved by induction for the positive integers. We leave proving the theorem for integers $a \leq 0$ as an exercise for the reader. □

Induction can be a useful way of proving divisibility, which is illustrated by the following example.

Example 4.3. Show that $3^{n+1} \mid 2^{3^n} + 1$ for all integers $n \geq 0$.

Proof. The statement applies for $n = 0$, since $3 \mid 2^{3^0} + 1 = 3$, which gives the base case. Next we assume $3^{k+1} \mid 2^{3^k} + 1$ for some integer $k \geq 0$, and use that assumption to prove $3^{k+2} \mid 2^{3^{k+1}} + 1$. We have,

$$2^{3^{k+1}} + 1 = (2^{3^k})^3 + 1^3.$$

Notice that this is a sum of two cubes and by using the well-known identity $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$, where $a = 2^{3^k}$ and $b = 1$, we get

$$\begin{aligned} 2^{3^{k+1}} + 1 &= (2^{3^k})^3 + 1^3 \\ &= (2^{3^k} + 1)((2^{3^k})^2 - 2^{3^k} + 1) \\ &= (2^{3^k} + 1)((2^{3^k} + 1)^2 - 3 \cdot 2^{3^k}). \end{aligned}$$

By the induction hypothesis $3^{k+1} \mid 2^{3^k} + 1$, so we have to show that $(2^{3^k} + 1)^2 - 3 \cdot 2^{3^k}$ is divisible by 3. By using the induction hypothesis $3^{k+1} \mid 2^{3^k} + 1$, then clearly $3 \mid (2^{3^k} + 1)^2$ and noting that $3 \mid 3 \cdot 2^{3^k}$, thus the difference is also divisible by 3. Which gives the desired result, $3^{k+2} \mid 2^{3^{k+1}} + 1$. By the principle of mathematical indcution we are done. □

Even some of the more difficult competitive number theory problems can be proved by mathematical induction. The following is a proof of problem 6 at the 1988 International Mathematical Olympiad, often deemed as one of the most difficult IMO problems. This proof was originally written by Dr J. Campbell (Canberra)[1].

[1]Campbell, John. *A Solution to 1988 IMO Question 6 (The Most Difficult Question Ever Set at an IMO).* Accessible at: `http://www.wfnmc.org/mc19882campbell.pdf`

Example 4.4 (IMO 1988)**.** Let a and b be positive integers such that $ab+1$ divides a^2+b^2. Show that

$$k = \frac{a^2+b^2}{ab+1}$$

is the square of an integer.

Proof. Some testing suggests $k = (GCD(a,b))^2$. We will prove this by induction on ab. By symmetry of a and b, we may assume $a \leq b$. We choose to include $a = 0$ as a possible value for a (which doesn't affect the proof, since we're proving for all greater integers). This makes the induction basis trivial, $ab = 0$ imply $a = 0$ and thereby $k = b^2 = (GCD(b,a))^2$. Now for our hypothesis we assume that the statement applies for all values of ab where $0 \leq ab < a_n b_n$. To prove it applies for $a_n b_n$, the next move is to look for an integer c which satisfies

$$0 \leq c < b_n$$

and

$$k = \frac{a_n^2+c^2}{a_n c+1} = \frac{a_n^2+b_n^2}{a_n b_n+1}$$

If such an integer exists, $0 \leq a_n c < a_n b_n$ and therefore $k = (GCD(a_n,c))^2$ by the induction hypothesis. To find c, we solve

$$k = \frac{a_n^2+c^2}{a_n c+1} = \frac{a_n^2+b_n^2}{a_n b_n+1}$$

The fractions are equal, so we may subtract the numerators and denominators, (note that we want $c < b_n \Rightarrow b_n - c \neq 0$)

$$k = \frac{b_n^2-c^2}{a_n b_n - a_n c} = \frac{(b_n+c)(b_n-c)}{a_n(b_n-c)} = \frac{b_n+c}{a_n}$$

or

$$c = a_n k - b_n$$

Notice that c is an integer and $GCD(a_n,c) = GCD(a_n,b_n)$, which means the proof will be finished if we can prove $0 \leq c < b_n$. First we prove the right-hand inequality

$$k = \frac{a_n^2+b_n^2}{a_n b_n+1} < \frac{a_n^2+b_n^2}{a_n b_n} = \frac{a_n}{b_n} + \frac{b_n}{a_n}$$

which imply (since $a_n \leq b_n$),

$$a_n k < \frac{a_n^2}{b_n} + b_n \leq \frac{b_n^2}{b_n} + b_n = 2b_n \Rightarrow a_n k - b_n < b_n \Rightarrow c < b_n$$

Next we prove $0 \leq c$. Because $k > 0$ and $a_n^2 + c^2 > 0$ we have

$$k = \frac{a_n^2+c^2}{a_n c+1} \Rightarrow a_n c + 1 > 0 \Rightarrow c > \frac{-1}{a_n} \geq -1$$

Which implies, since c is an integer, $c \geq 0$. For every b_n there exists a $c = a_n k - b_n$ satisfying the stated criteria, and the statement is proven by induction. ☐

4.1 Problems

Problem 4.1. Prove $6 \mid n^3 - n$ for all $n \in \mathbb{N}$.

Problem 4.2. Prove that $6^n - 1$ is divisible by 5 for all $n \in \mathbb{N}$.

Problem 4.3. Prove that $\frac{(2n)!}{n!} = 2^n \cdot 1 \cdot 3 \cdot 5 \cdots (2n-1)$ for all $n \in \mathbb{N}$.

Problem 4.4. Show that all numbers of the form

$$1003, 10013, 100113, 1001113, 10011113, \ldots$$

are divisible by 17.

5 Graph theory

Graph theory is the theory about graphs, that is structures of vertices joined together with edges. Induction is often useful for proving properties true for certain graphs.

Example 5.1. Show that in a *tree*[2] with v vertices there are exactly $v - 1$ edges.

Proof. Let the induction hypothesis be that every tree with $v, v-1, v-2, \ldots, 1$ vertices respectively contains $v-1, v-2, v-3, \ldots, 0$ edges. It is easy to verify that all trees with 1 vertex have 0 edges, and that all trees with 2 vertices have 1 edge. Now consider an arbitrary tree with $v+1$ vertices. The induction step is to prove that the tree has v edges.

We know that in a tree there is exactly one path between every pair of vertices. Hence, removing the edge between two vertices yields two separate components with n and $v + 1 - n$ vertices each. It is evident that we cannot create new paths between vertices by removing an edge, thus both components are trees. But $1 \leq n < v$, so according to our induction hypothesis they have $n - 1$ and $v - n$ edges respectively. However we removed one edge from the original graph. Consequently the tree with v edges had $n - 1 + v - n + 1 = v$ edges, so the induction hypothesis holds.

□

Example 5.2. Show that a *simple graph*[3] with $2n$ vertices that contains no triangle has at most n^2 edges. A *triangle* is a set of 3 vertices such that any two of them are connected by an edge of the graph.

Proof. The problem is equivalent to proving that if there are more than n^2 edges in a graph with $2n$ vertices, there has to be a triangle. We can assume that there are $n^2 + 1$ edges, since more edges only would result in more triangles.

[2] `https://en.wikipedia.org/wiki/Tree_(graph_theory)`
[3] `https://mathworld.wolfram.com/SimpleGraph.html`

Our induction hypothesis is that there exist at least one triangle in a graph with $2n$ vertices and n^2+1 edges. When $n = 1$ there are two vertices, which cannot have more than one edge connecting them. Since $1 < n^2+1$ the assumption holds for the base case $n = 1$.

Consider a graph with $2(n+1)$ vertices and $(n+1)^2+1$ edges. Suppose that there's an edge between the vertices u and v. The *subgraph* not containing u and v have $2n$ vertices. Notice that this graph either has at least $n^2 + 1$ edges or at most n^2 edges. In the former case there must be a triangle according to the induction hypothesis. If there instead are at most n^2 edges, there will be at least $(n + 1)^2 + 1 - 1 - n^2 = 2n + 1$ edges between u or v and the sub graph. But there are only $2n$ vertices in the subgraph, so by the pigeonhole principle u and v must have edges connecting to the same vertex in the subgraph, hence forming a triangle.

□

5.1 Problems

Problem 5.1. Show that a complete graph with v vertices has $\frac{v(v-1)}{2}$ edges.

Problem 5.2 (Euler's formula)**.** Prove that in connected planar[4] graphs with V vertices, E edges and F faces (regions bounded by edges), the following formula is true

$$V - E + F = 2.$$

When using Euler's formula, we always include the infinitely large face on the outside.

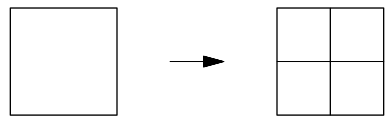
6 Combinatorial geometry

In this section we will look at some problems that contains the objects from geometry and the ideas from combinatorics. There are a lot of interesting ideas that can be combined with induction to solve these types of problems. Usually we are dealing with points or lines in the plane and we want to investigate some property of it. A common idea that we will apply later on is to remove some points such that we can apply the induction hypothesis on the remaining points and after that add back the missing points and try to find something that could potentially help us. In the introduction we talked about splitting up the *inductive step* but we haven't seen any problems where that could be useful yet. Let's do that now. Consider the following problem.

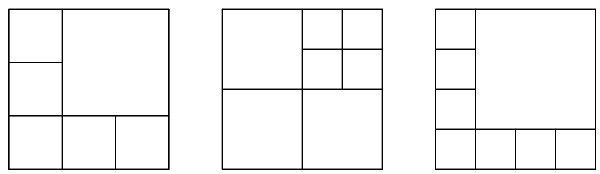
Example 6.1. Prove that each square can be divided into n smaller squares for $n \geq 6$.

Proof. Let $P(n)$ be the statement that each square can be divided into n smaller squares. Notice that a single square can easily be divided into 4 smaller squares thus adding 3 new squares.

[4] `https://en.wikipedia.org/wiki/Planar_graph`



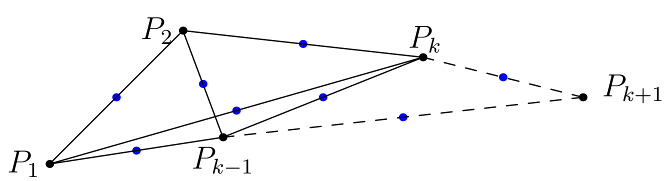
By this observation we know that if $P(k)$ is true for some integer $k \geq 6$, then $P(k+3)$ is also true. Hence we need 3 base cases to complete the proof. We need to find constructions for $n = 6, 7, 8$, and that is relatively easy to do.



By the principle of mathematical induction we are done. □

Example 6.2 (SMT 2008, P6)**.** Let $Q_1, Q_2, ..., Q_n$ be $n \geq 2$ distinct points in the plane. We colour the midpoints of all possible segments between the points blue ($Q_iQ_j, 1 \leq i < j \leq n$). What is the least possible number of distinct blue points?

Proof. Some testing suggests that the answer is $2n-3$ blue points. We will show that this is the lower bound and leave the construction as an exercise for the reader. We proceed by induction. As usual we check if the base case is true. If $n = 2$ then we need at least $2 \cdot 2 - 3 = 1$ blue point which is obviously true. As the induction hypothesis we assume that the statement is true for $n = k$ for some integer $k \geq 2$. Let us consider any arrangement of $k+1$ points in the plane. By using a perturbation argument we can rotate the plane such that the points have no common x-coordinate (why can we do this?). Label the points $P_1, P_2, \ldots, P_{k+1}$ from left to right. The trick is to remove the point P_{k+1} so that we can use the induction hypothesis on the remaining k points. When we add back the point P_{k+1} we have to find at least $(2(k+1)-3)-(2k-3) = 2$ new blue points. We can easily do this by noting that the midpoint of $P_{k+1}P_k$ and $P_{k+1}P_{k-1}$ must be new, since these two midpoints are to the right of the other midpoints.



□

7 Problems

In the final section we have collected various problems which can be solved using mathematical induction. Have fun solving!

Problem 7.1. Show that $n^n \geq (n+1)^{(n-1)}$ for all positive integers n.

Problem 7.2. Show that

$$1^2 + 2^2 + 3^2 + ... + n^2 = \frac{2n^3 + 3n^2 + n}{6}.$$

Problem 7.3. A sequence $a_1, a_2, ...$ is defined by $a_1 = 0$ and $a_{n+1} = 2a_n^2 - 5a_n + 4$. What is a_{31415}?

Problem 7.4 (*Formula for geometric sum*)**.** Let $n \in \mathbb{N}$, $a \in \mathbb{R}$ and $a \neq 1$. Prove

$$\sum_{m=0}^{n} a^m = \frac{a^{n+1} - 1}{a - 1}$$

Problem 7.5 (Russia, 1993)**.** The integers from 1 to n are written in a line in some order. The following operation is performed with this line: if the first number is k then the first k numbers are rewritten in reverse order. Prove that after some finite number of these operations, the first number in the line of numbers will be 1.

Problem 7.6 (*British Mathematical Olympiad Round 1 2010/11, P4*)**.** Isaac has a large supply of counters, and places one in each of the 1×1 squares of an 8×8 chessboard. Each counter is either red, white or blue. A particular pattern of coloured counters is called an *arrangement*. Determine whether there are more arrangements which contain an even number of red counters or more arrangements which contain an odd number of red counters. *Note that* 0 *is an even number.*