Points and Lines in Combinatorial Geometry

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Contents

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1 Introduction

Combinatorial geometry is a blend of combinatorics and geometry. It deals with combinatorial problems that have a geometric setting and combines techniques from combinatorics, geometry, graph theory, number theory and more. Problems from combinatorial geometry appear frequently in mathematical olympiads and our aim is to help you learn some basic instruments for how to solve them. We will deal mostly with combinatorial geometry that revolves around lines and points in the plane, but we note that combinatorial geometry can also deal with other geometrical objects and higher dimensions.

Because of the great variety of useful techniques related to combinatorial geometry, we have decided not to cover the following techniques: induction, invariants and monovariants. For interested readers we recommend the introductory articles about the Invariant Principle[3] and Induction[4] on the Brilliant Math & Science Wiki. Knowledge of these subjects is not required for any parts of this text, but may be useful in other contexts.

1.1 Conventions in Combinatorial Geometry

Often in combinatorial geometry problems, you are given a set of points or a set of lines with some form of constraint. The two most common constraints are that no three points are *collinear* and that no three lines are *concurrent*. A number of points are *collinear* if there exists a line that passes through all of them. Similarly, a number of lines are *concurrent* if they share a common point. Another common constraint in combinatorial geometry problems is that the points or lines are in *general position*. This constraint is a bit arbitrary in the sense that it means that there are no 'special' cases - which must be interpreted according to the context. In combinatorial geometry, this means that no three points are *collinear*, no three lines are *concurrent*, that no two lines are parallel, and that no two points or lines coincide. Sometimes it also means that no four points are concyclic, i.e lie on a circle, but this will not be relevant for the problems in this text.

2 Points, Lines and Faces

A common task in combinatorial geometry is to count points and lines. In this chapter we will show some effective ways in which this can be done.

Example 2.0.1. *We have n lines in general position. What is the number of intersection points expressed in n?*

Solution. We begin by noticing that, since all lines are in *general position*, every pair of lines must intersect, as no pair of lines are parallel. Additionally, this intersection point is unique as no third line can pass through it (otherwise we would have three *concurrent* lines which is not allowed as the lines are in *general position*). Hence the number of intersection points is equal to the number of possible pairs of n lines.

Counting the number of possible pairs of n lines is easy. We have n choices for the first line and $n - 1$ for the second. Hence we have $n(n - 1)$possible ordered pairs of lines. As the order doesn't matter, we divide this number by 2 as there are 2 possible orderings for each pair of lines. Hence the number of possible pairs of n lines is $\frac{n(n-1)}{2}$ which was equal to the number of intersection points. □

Exercise 2.0.2. *Find a construction with* n *lines (not necessarily in general position) such that the number of intersection points* (I) *satisfy:*

(a) $I = 1$
(b) $I = n - 1$

[3] "Invariant Principle." *Brilliant Math & Science Wiki*, brilliant.org/wiki/invariant-principle-definition/.
[4] "Induction." *Brilliant Math & Science Wiki*, brilliant.org/wiki/induction/.

(c) $I = n$

2.1 Binomial coefficients

In combinatorics, binomial coefficients $\binom{n}{k}$ denote the number of ways to choose k objects from a set of n objects (where every object may only be chosen once and each object is unique). Binomial coefficients have a special notation $\binom{n}{k}$ which is read as "*n choose k*". A useful property of the binomial coefficient is the recurrence relation below:

Exercise 2.1.1. *Prove that* $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$ *using the definition of binomial coefficients.*

Theorem 2.1.2. $\binom{n}{k} = \frac{n!}{k!(n-k)!}$. (Where $a!$, spoken "a factorial", is $a \cdot (a-1) \cdot (a-2) \cdot ... \cdot (2) \cdot 1$)

Proof. We will use a generalized version of the argument used in *Example 2.0.1* in this solution. Consider choosing an ordered subset of size k from a set of size$n$. We have n choices for the first object, $n-1$ for the second, $n-2$ for the third and so on until we have $n-(k-1)$ choices for the kth object. Hence there are $n \cdot (n-1) \cdot ... \cdot (n-(k-1)) = \frac{n!}{(n-k)!}$ possible ordered subsets of size k. To compute $\binom{n}{k}$ we only need to divide by the number of ways to permute a set of size k, which is equal to the number of ordered subsets of size k from a set of size k, i.e. $k \cdot (k-1) \cdot ... \cdot 1 = k!$. Hence $\binom{n}{k} = \frac{n!}{k!(n-k)!}$. □

Exercise 2.1.3. *Let S be a set of n points in general position. How many triangles with vertices in S are there?*

2.2 Euler's Characteristic Formula

Sometimes, it can be a good idea to convert a construction into a graph in order to enable applications of techniques from graph theory. A graph consists of *vertices* and *edges*. An *edge* connects two *vertices*. A *planar graph* is a special type of graph. It is a *connected* finite graph which can be drawn on the plane without any intersecting *edges* (A graph is *connected* if there exists a path traversing only *edges* between every pair of *vertices*). These *edges* are not necessarily straight lines. A *face* is an area enclosed completely by *edges* and may be infinitely large. In a graph V, E and F are used to denote the number of *vertices*, *edges* and *faces* respectively. Note that the number of *faces* also includes the infinite region outside/surrounding the graph.

Theorem 2.2.1. (Euler's Characteristic Formula) *In a planar graph the equality* $V - E + F = 2$ *holds.*

Example 2.2.2. *How many regions do* n *lines in general position split the plane into?*

Solution. We begin by converting our construction to a planar graph. Let all intersection points between the lines become *vertices* and let every line segment between two intersection points become an *edge*. Let the line segments going off to infinity bend to meet at a point (and hence become finite in length). This transformation does not create nor remove any regions (*faces)*. Hence: $V = 1 + no.\ of\ intersection\ points = 1 + \binom{n}{2}$.

Every line is divided by $n-1$ intersection points and 2 endpoints into n segments (*edges*). Hence $E = n^2$.

Applying Euler's characteristic formula and rearranging, we get that: $F = \frac{n^2+n+2}{2}$. □

Exercise 2.2.3. (3Blue1Brown) *If you take* n *points on a circle, then connect every pair of them with a line such that no 3 lines are concurrent, how many regions do these lines cut the circle into?*

3 Pigeonhole Principle

The pigeonhole principle, also known as Dirchlet's box principle, is sometimes considered as something very obvious. However, whether it's obvious or not, it is a very useful tool when solving problems in combinatorics and combinatorial geometry. The challenge one faces while solving problems using this principle is finding the correct "pigeons" (objects) and the correct "pigeonholes" (containers). Therefore, this chapter will mainly consist of practice problems. Note that the "pigeons" and "pigeonholes" are not necessarily discrete (see *Theorems 3.2 & 3.3*).

Theorem 3.1. (Dirchlet's box principle) *If* $nk + 1$*, or more, pigeons are placed into* n *holes, then there exists a hole which contains* $k + 1$ *or more pigeons.* $\forall\, n, k \in \mathbb{N}$.

Theorem 3.2. *If more than* nk *units of fluid are poured into* n *containers, then there exists a container with more than* k *units of fluid.* $\forall\, n \in \mathbb{N},\ k \in \mathbb{R}^{+}$.

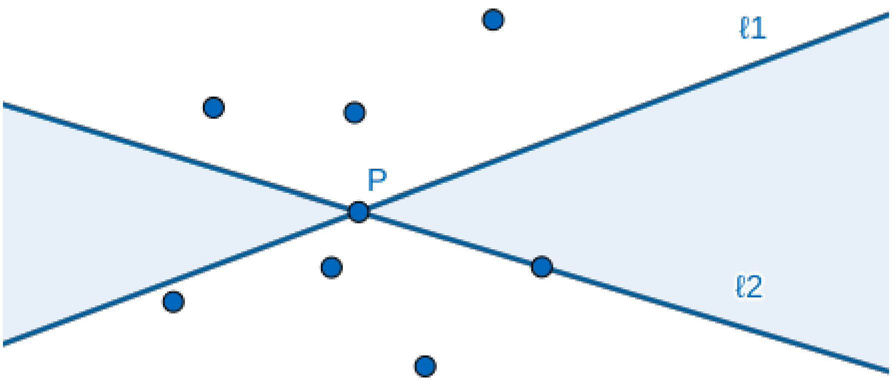
Theorem 3.3. *If the integral of a function* f *is greater than* nk*, when evaluated over an interval of size* n*, then* f *must attain a value greater than* k *somewhere on this interval.*$\forall n, k \in \mathbb{R}^{+}$.

Exercise 3.4. Several *arcs, the sum of whose lengths is greater than 2π, lie on a circle of radius 1. Prove that at least two of the arcs share a common point.*

Example 3.5. *Show that among any 5 points inside an equilateral triangle of unit side length, there are 2 points which are at most 1/2 units apart.*

Solution. Divide the unit triangle into 4 equilateral triangles with the sides of 1/2 units by connecting the midpoints of the large triangle. We now have 4 holes (small triangles) and 5 pigeons (points), which means one of the triangles must contain at least two points. In an equilateral triangle, the greatest distance between two points is the side length, hence there are two points which are at most 1/2 units apart. □

Exercise 3.6. *Two lines* ℓ_1 *and* ℓ_2 *intersect at* p *at an angle of* $\frac{180°}{n}$*, and create two opposing sectors of the plane. Given a set S of* $n + 1$ *points in the plane (including* p*), prove that two such lines* ℓ_1 *and* ℓ_2 *can always be chosen such that the two opposing sectors of the plane they create contain no points of S. (Points of S CAN lie on* ℓ_1 *and* ℓ_2*)*



Exercise 3.7. *There are 6 points in a 3 by 4 rectangle. Prove that there are two points whose distance does not exceed* $\sqrt{5}$.

Exercise 3.8. (SMT-finals 2008) *A convex n-gon has angles a_1, a_2, a_3,..., a_n (in degrees), where every a_k is a positive whole number divisible by* 36 *($k \in \{1, 2, ..., n\}$). Show that if $n > 5$, then two of the n-gon's sides must be parallel.*

Exercise 3.9. (Cut The Knot) *Prove that among any 9 points in a triangle with area 1, there are three points that form a triangle of area not exceeding 1/4. Prove that this is also true if instead of 9 points, there are only 7.*

Exercise 3.10. (Based on a problem from the book "Problem-Solving Strategies" by Arthur Engel) *Consider a line ℓ in the plane and a point p_0 on ℓ. One places 49 points on ℓ(all on the same side of p_0) such that all of their distances to p_0 are integers. Given that no two points coincide and that the distance from every point to p_0 is at most 76 units, show that there exist two points on the line (not including p_0) which are exactly 21 units apart.*

4 Pick's Theorem

A *lattice point* is a point in the cartesian coordinate system (x- and y-axes are perpendicular) where the x- and y -coordinates are integers. One famous and useful theorem using *lattice points* is Pick's theorem, which expresses the area of a polygon with all of its vertices on *lattice points*. The theorem expresses the area in terms of the number of *lattice points* in the *interior* (I) and on the *boundary* (B) of the polygon.

Theorem 4.1. (Pick's Theorem) *For any polygon with all vertices on lattice points in a lattice grid, its area A can be expressed as $A = I + \frac{B}{2} - 1$, where I is the number of lattice points inside of the polygon, and B the number of lattice points on the boundary of the polygon.*

Exercise 4.2. *Prove that Pick's theorem holds for all axis-parallel rectangles.*

Example 4.3. *Prove that Pick's theorem holds for all right-angled triangles with axis-parallel short legs.*

Solution. Assume that Pick's theorem holds for all axis-parallel rectangles. Denote by T a right-angled triangle with axis-parallel short legs as in *Figure 4.1*. Place a congruent right angled triangle along the hypotenuse of the original right-angled triangle to form an axis-parallel rectangle, which we denote by R (see *Figure 4.1.*). By symmetry, these two triangles have equal area, number of *interior* points and number of *boundary* points. Looking at *Figure 4.1*, we notice the following: (Note that it is useful to regard A, B and I as functions).

- $2 \cdot A(T) = A(R)$
- $B(T) = n + m + k - 3$
- $B(R) = 2(n + m - 2) = 2 \cdot (B(T) - k + 1)$
- $2 \cdot I(T) = I(R) - (k - 2)$

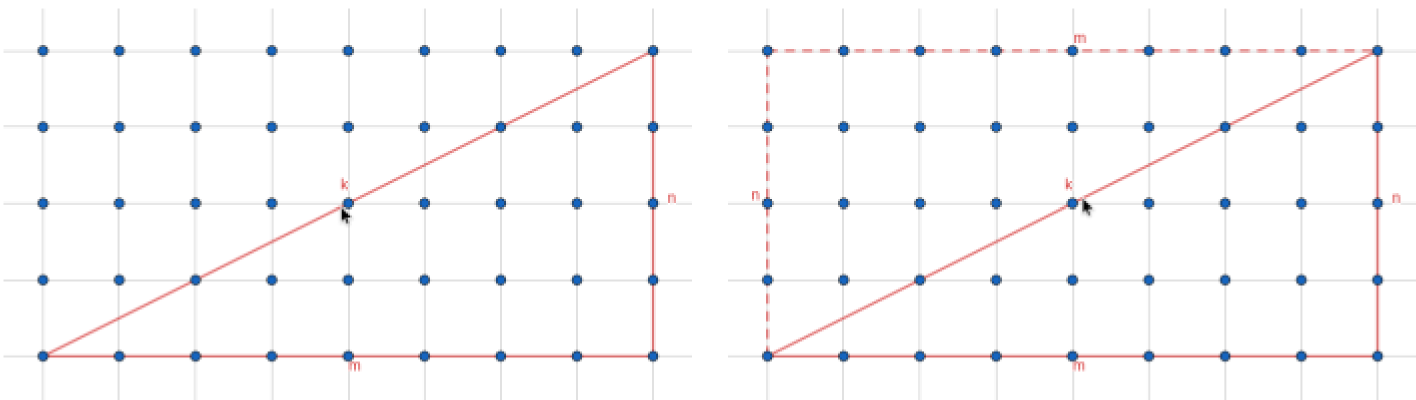


Figure 4.1. m,n and k represent the number of lattice points (including vertices), not length.

Using these equations and Pick's theorem for axis-parallel rectangles, we get that:

$$2 \cdot A(T) = A(R) = I(R) + \frac{1}{2}B(R) - 1 = 2 \cdot I(T) + (k-2) + \frac{2\cdot(B(T)-k+1)}{2} - 1.$$

$$2 \cdot A(T) = 2 \cdot I(T) + B(T) + (k-2) + (-k+1) - 1 = 2 \cdot I(T) + B(T) - 2.$$

$A(T) = I(T) + \frac{1}{2}B(T) - 1$. Which is precisely Pick's theorem. □

Exercise 4.4. *Show that all rectangles with vertices on lattice points have integer area.*

Example 4.5. *Prove that Pick's theorem holds for all triangles. (Except the type from Example 4.3.)*

Solution. Let T be a triangle with vertices on *lattice points*. Construct a rectangle R such that: R's vertices are on *lattice points*, R contains T and R's boundary contains a maximal amount of T's vertices (the amount of vertices will be 2 or 3). Afterwards, if one of the sides of T is axis-parallel, then let R extend a distance beyond this side, so R's boundary does not contain a full side of T (this may reduce the number vertices of T on R's boundary to 2).

Case 1: (R's boundary contains all 3 of T's vertices) Then T divides R in 3 axes-parallel right-angled triangles or rectangles with vertices on *lattice points* (see *Figure 4.2*), and let these be A, B and C. Looking at the construction, we notice the following:

- $I(R) = I(A) + I(B) + I(C) + I(T) + (B(T) - 3)$
- $B(R) = B(A) + B(B) + B(C) - B(T)$
- $A(R) = A(A) + A(B) + A(C) + A(T)$

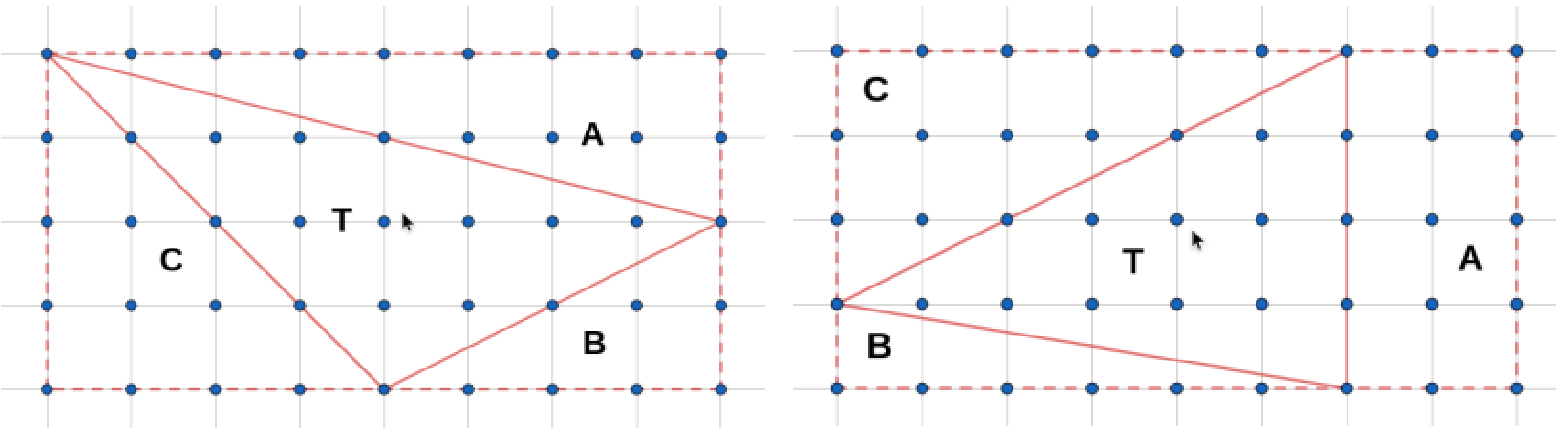


Figure 4.2. A, B, C and T denote triangles and rectangles whose union is the rectangle R.

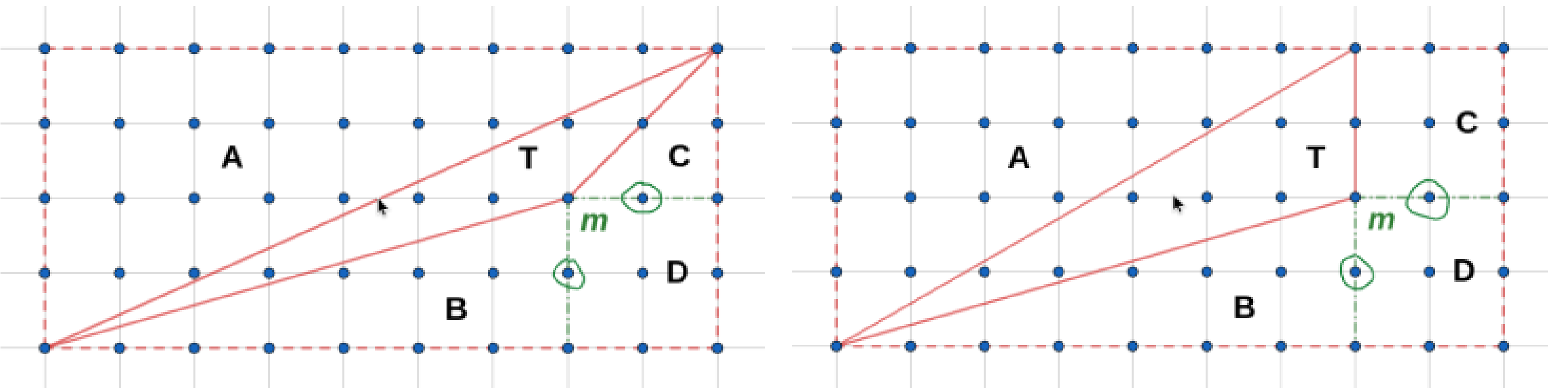


Figure 4.3. A, B, C, D and T denote triangles and rectangles whose union is the rectangle R. m denotes the number of lattice points (excluding vertices) that lie on any common boundaries between A, B, C and D (these are circled in green for clarity).

Case 2: (R's boundary contains only 2 of T's vertices) Then R can be divided into 4 axes-parallel right-angled triangles and rectangles with vertices on *lattice points*, specifically the boundary of R or the vertices of T. Let these be A, B, C and D (see *Figure 4.3.*). Let m be the number of lattice points (possibly none) on the shared boundaries of A, B, C and D (excluding their vertices). Looking at the picture, we notice that:

- $I(R) = I(A) + I(B) + I(C) + I(D) + I(T) + B(T) + m - 2$
- $B(R) = B(A) + B(B) + B(C) + B(D) - 2m - B(T) - 4$
- $A(R) = A(A) + A(B) + A(C) + A(D) + A(T)$

In both *Case 1* and *Case 2*, denote by S_I the sum of all the interior points of A, B, C (and D in *Case 2*), by S_B the sum of all their boundary points, and by S_A the sum of their areas. Furthermore, using Pick's theorem for axis-parallel right-angled triangles and rectangles, we have that:

$$A(X) = I(X) + \frac{1}{2}B(X) - 1,\ X \in \{A, B, C, D, R\}$$

Combining all of these equations and denotations, we have that:

Case 1: $A(R) - A(T) = S_A = S_I + \frac{1}{2}S_B - 3 = I(R) - I(T) - B(T) + 3 + \frac{1}{2}(B(R) + B(T)) - 3$

$A(R) - A(T) = I(R) + \frac{1}{2}B(R) - 1 + 1 - I(T) - \frac{1}{2}B(T)$

(By Pick's theorem for A, B & C)

$A(R) - A(T) = A(R) + 1 - I(T) - \frac{1}{2}B(T)$ (By Pick's theorem for R)

$- A(T) = - I(T) - \frac{1}{2}B(T) + 1$

Case 2: $A(R) - A(T) = S_A = S_I + \frac{1}{2}S_B - 4$

$A(R) - A(T) = I(R) - I(T) - B(T) - m + 2 + \frac{1}{2}(B(R) + B(T) + 2m + 4) - 4$

$A(R) - A(T) = I(R) + \frac{1}{2}B(R) - 1 + 1 + (m - m + 4 - 4) - I(T) - \frac{1}{2}B(T)$

(By Pick's theorem for A, B, C & D)

$A(R) - A(T) = A(R) + 1 - I(T) - \frac{1}{2}B(T)$ (By Pick's theorem for R)

$- A(T) = - I(T) - \frac{1}{2}B(T) + 1$

Multiplying by -1, we get that: $A(T) = I(T) + \frac{1}{2}B(T) - 1$, which is exactly Pick's Theorem. □

Exercise 4.6. *Prove that there does not exist an equilateral triangle with all 3 vertices on lattice points.*

Problem 4.7. *Prove that if Pick's theorem holds for all n-gons, then it holds for all $n + 1$-gons. (Use the property that every polygon can be triangulated[5])*

5 Geometric Constructions

With any problem in combinatorial geometry, the geometrical properties of the construction will be essential for the solution. In this chapter we will provide a multitude of combinatorial geometry problems whose solutions use geometric facts related to the construction. Most of these facts are unique to each problem, but a lot of them

[5] Triangulation is the process of dividing a n-gon into $n-2$ triangles with the same vertices as the n-gon. This process can amongst other things prove that the inner angle sum of a n-gon is: $(n-2)\times180°$.

are related to: the triangle inequality, area, perimeter, angles and extreme positions (see chapter 6). These facts can then be used to find/put some constraint on the construction, which can further lead to a solution.

Theorem 5.1. (Triangle Inequality) *In any non-degenerate triangle $\triangle ABC$ (a triangle with a non-zero area), the inequalities $|AB| < |BC| + |AC|$, $|BC| < |AB| + |AC|$ and $|AC| < |AB| + |BC|$ hold.*

Example 5.2. *Let $\square ABCD$ be a convex quadrilateral, prove that $|AC| + |BD| > |AB| + |CD|$.*

Solution. Begin by denoting the intersection of the diagonals AC and BD by O. Since $\square ABCD$ is convex, O will always be inside the quadrilateral. Using the triangle inequality on the triangles $\triangle ABO$ and $\triangle CDO$ gives the following:

Triangle inequality in $\triangle ABO$: $|AO| + |BO| > |AB|$
Triangle inequality in $\triangle CDO$: $|CO| + |DO| > |CD|$
$\Rightarrow |AB| + |CD| < |AO| + |BO| + |CO| + |DO| = |AC| + |BD|$ □

Exercise 5.3. *Let $\square ABCD$ be a convex quadrilateral with $|AD| < |BC|$ and let P be a point in its interior. Let x and y be the distances[6] from P to AB and CD, respectively. Show that $x + y \leq |BC|$.*

Example 5.4. *In a set of points, the triangle with the smallest area is given. Find the region where the other points of the set could be.*

Solution. Let $\triangle ABC$ be the triangle with the smallest area. Then no other point X should be able to create a triangle together with 2 of the vertices of $\triangle ABC$, and with area smaller than $\triangle ABC$. Let's consider where X needs to be so that $\triangle XBC$ has an area larger than $\triangle ABC$. As area of a triangle is it's base times its' height all over 2, (and $\triangle XBC$ and $\triangle ABC$ share base BC) the height of $\triangle XBC$ from X to BC must be larger than the height from A to BC (h_a). Hence X cannot lie on the strip of width $2 \cdot h_a$ parallel to BC with BC in its middle. Similar constraints apply for X in relation to the sides AB and AC. Hence we have determined the zones where X must be if we are to cause a contradiction, and hence X can be everywhere but in these zones (see *Figure 5.1.*). □

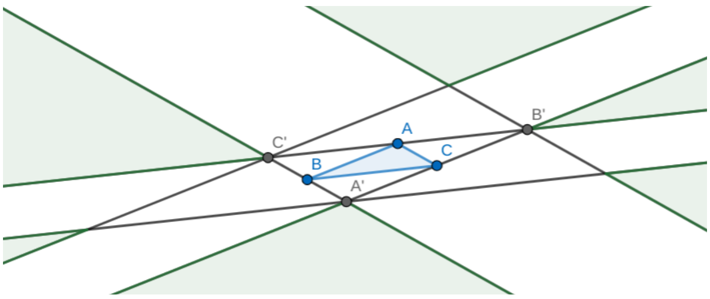


Figure 5.1. Green regions indicate where the other points can be if the blue triangle is the triangle with the smallest area.

Exercise 5.5. (Cut The Knot) *A finite number of points in the plane have the property that the area of a triangle formed by any three of them is at most 1. Prove that there exists a triangle of area not more than 4 that contains all points.*

Exercise 5.6. *In a set of points, the pair of points A and B are the closest/furthest away from each other. Prove that all other points in the set must lie on the outside/inside (respectively) of the circle with centre at the midpoint of segment $\overline{AB}$ and with diameter $\sqrt{3} \cdot |AB|$.*

Example 5.7. *Given that the length of the the longest side of a triangle is x, find the maximum area of that triangle in terms of x.*

[6] The distance from a point to a line is defined to be the distance from the point itself to the orthogonal projection of the point onto the line. This is the shortest distance between the point and any point on the line.

Solution. Denote the triangle by $\triangle ABC$ with longest side AB of length x. From Exercise 5.5 we know that C can therefore only exist inside a circle with radius $\frac{\sqrt{3}x}{2}$ with centre M, the midpoint of AB. However, this means that C is at most distance $\frac{\sqrt{3}x}{2}$ from AB and therefore $\triangle ABC$ has a height of at most $\frac{\sqrt{3}x}{2}$. This gives that the maximum area of $\triangle ABC$ is $\frac{\sqrt{3}x^2}{4}$. To verify that such a triangle exists, consider the case when $\triangle ABC$ is equilateral. □

There are very many geometrical properties that could be useful in combinatorial geometry problems, all of which cannot be covered in this text. However, some are covered in the following exercises:

Exercise 5.8. (Finnish High School Competition 2007) *There are five points in the plane, no three of which are collinear. Show that some four of these points are the vertices of a convex quadrilateral.*

Exercise 5.9. *Using the result from the previous problem, prove that there exists a non-acute (right or obtuse) angled triangle with vertices in any set of five points in the plane in general position.*

Exercise 5.10. *Prove that there exists a non-acute (right or obtuse) angled triangle with vertices in any set of four points in the plane in general position.*

Exercise 5.11. (Ung Vetenskapssport Matematikturné 2021) *Given is a triangle in the plane with vertices at coordinates (0,0), (1,5) and (3,3). At every move, we may move one vertex, lets say A, to another point A' such that AA' is parallel to the line through the other two vertices. Show that the triangle's vertices cannot end up having coordinates (1,3), (4,1) and (5,4) at the same time after finitely many moves.*

Exercise 5.12. *Prove that a triangle inside a parallelogram of area 2 must have an area not greater than 1.*

Exercise 5.13. (Adapted from IMO Winter Camp 2009) *There are five points in the plane. Prove that at least four of them can be selected such that no three of these are vertices of an equilateral triangle.*

6 Extremal Arguments

An Extremal Argument (also called Extremal Principle) is a method that can often be used in combinatorial geometry to prove that there exists a construction, or an object inside a construction, that satisfies some wanted condition. An example of such a problem would be proving the Sylvester-Gallai Theorem:

Theorem 6.1: (Sylvester-Gallai Theorem) *For any finite non-collinear set of points in the plane there is a line passing through exactly two of them.*

In combinatorial geometry problems, there are often infinitely many constructions, and objects inside those constructions, which we need to consider. However, to limit the possibilities and gain more information we can always consider the "extreme" construction or object, with a property that is the smallest/largest for all such objects or constructions. Depending on the property we consider, this can give us a lot of information. From this we can either try to find a construction that satisfies our wanted condition directly, or we can instead assume that our wanted condition does not hold for some construction and then try to find a contradiction to that assumption, often by constructing a more "extreme" construction or object.

Examples of "extreme" properties that could be used for Extremal Arguments include:

- The path through a set of points that has a minimal or maximal length,
- The triangle with minimal/maximal perimeter, area, height or radius of the circumscribed circle,
- The quadrilateral (or other polygon) with the most/fewest points inside,
- The two points closest/furthest away from each other, or

- The points furthest in some direction.

Proof. Denote by S any finite non-collinear set of points in the plane. When there are only 2 points in S, the result is obvious. Let us instead consider the case when we have 3 or more points in S.

Let us consider a pair of a point in S and a line passing through two other points in S. (Such a pair will always exist since all points in S are not collinear, and there are at least 3 points in S). Let us consider the "extreme" such pair (P, ℓ) such that the distance between P and ℓ is minimal.

Now let us assume that the Sylvester-Gallai Theorem does not hold for S, and try to find a contradiction. This implies that, for our line ℓ there are at least 3 points in S that it passes through, of which at least 2 are on the same side of, or on, the normal PP' from ℓ to P. Of these points, let A be the one furthest away from the normal, and B be the one closest to the normal.

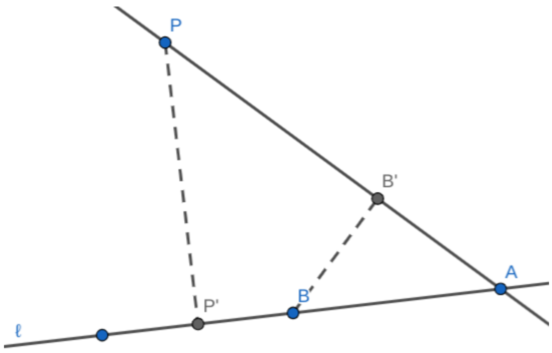


Figure 6.1. Blue points A, B, P in S are marked, as well as the normals PP' to ℓ and BB' to AP.

Let us label BB' the normal from AP to B, and notice that $\triangle APP' \sim \triangle ABB'$ because $\angle PAP' = \angle BAB'$ and $\angle PP'A = \angle BB'A = 90°$. This gives us that $\frac{PP'}{AP} = \frac{BB'}{AB}$, and since $AP > AP' \geq AB$ the inequality $PP' > BB'$ follows (see *Figure 6.1.*).

However, this implies that the distance from B to the line AP is strictly smaller than the distance from P to ℓ. Hence we have reached a contradiction: that our pair (P, ℓ) is not the "extreme" such pair, since (B, AP) is more "extreme". Therefore our assumption that the Sylvester-Gallai Theorem does not hold for some finite non-collinear set of points in the plane, is false. Hence the Sylvester-Gallai Theorem is true. □

Exercise 6.2. *Let S be a set of red and blue points (at least 2 of each) in the plane in general position. Prove that there exists a triangle T with vertices in S such that not all vertices of T are the same color and no other points of S lie inside T.*

It is important to realise that an Extremal Argument does NOT have to end in a contradiction, but can instead be slightly tweaked to construct a suitable construction satisfying the wanted condition.

Example 6.3. (Adapted from CMO 2017) *Let S be a set of points in the plane, such that any triangle with vertices in S has area of at most A. Prove that there exists a line ℓ in the plane such that the distance to ℓ from any point in S is at most $\sqrt{A}$.*

Solution. Let us observe the longest segment PQ $(P, Q \in S)$ and let $|PQ| = d$. We therefore know that, for any other point $M \in S$ the distance m of the perpendicular normal from M to PQ must be such that the area of $\triangle PQM$ is $\leq A$. Therefore we have that $\frac{m \times d}{2} \leq A \Leftrightarrow m \leq \frac{2A}{d}$.

If $d \geq 2\sqrt{A}$ we are done, since we can choose ℓ to be PQ and have that the distance m from any point $M \in S$ to ℓ is $m \leq \frac{2A}{d} \leq \sqrt{A}$. If instead $d < 2\sqrt{A}$ we can choose ℓ to be the perpendicular bisector (normal from the midpoint) of PQ and notice that P and Q are the two points furthest away from ℓ in S at a distance of $\frac{d}{2} < \frac{2\sqrt{A}}{2} = \sqrt{A}$. Note that, if there is a point $M \in S$ such that the distance m from ℓ to M is $m > \frac{d}{2}$, then either $PM \geq m + \frac{d}{2} > d = PQ$ or $QM \geq m + \frac{d}{2} > d = PQ$ which contradicts the fact that PQ is the longest segment between two points in S. □

Exercise 6.4. (Canadian IMO Training 2019) *Given $2n$ points in a plane with no three collinear, with n red points and n blue points, prove that there exists a pairing of the red and blue points such that the segments joining each pair are pairwise non-intersecting.*

7 "Sweeping" Lines

The "sweeping technique" is a useful trick that occurs in many combinatorial geometry problems. A "sweeping line" is an imaginary line that is swept or moved across the plane, stopping at some special points. We will demonstrate further how we can use a "sweeping line" in the example below.

Example 7.0.1. *There are $2n$ points in the plane. Show that there exists a line ℓ which divides the plane into two regions with n points in each.*

Solution. Let us define the set of $2n$ points in the plane as S for clarity. Notice that there are only finitely many points in S, therefore only finitely many lines connecting two points in S, and therefore only finitely many slopes of lines connecting two points in S. Let us now choose ℓ' to be any line in the plane with a slope that does not occur in the finite set of slopes of lines connecting two points in S. Let us move ℓ' "away" such that all the points of S are on one side of it.

Now we begin our "sweeping technique" with ℓ' as our "sweeping line" without changing the slope of ℓ'. As we "sweep" ℓ' across the plane towards the points of S we will eventually hit a point in S, which we pass over. Note that ℓ' cannot hit two different points in S at the same time, as this would mean that the slope of ℓ' was not such that there does not exist a line connecting two points in S with that slope. This means that we can "sweep" ℓ' over exactly n points of S, stop, and then our line ℓ' will satisfy the wanted condition for ℓ. □

Using the visualisation of moving lines across the plane, we can surprisingly solve many different problems. The line doesn't have to move in a linear direction either, it can rotate and change movements. However, the crux is finding when and where to use this technique effectively.

Exercise 7.0.2. *There are Cn points in the plane in general position. Show that there exists $C - 1$ parallel lines $\ell_1, \ell_2, \ldots, \ell_{C-1}$ which split the plane into C regions with n points in each region.*

Exercise 7.0.3. (Finnish National High School Mathematics Competition 2010) *Let S be a non-empty subset of a plane. We say that the point P can be* seen *from A if every point from the line segment AP belongs to S. Further, the set S can be* seen *from A if every point of S can be* seen *from A. Suppose that S can be* seen *from A, B and C where ABC is a triangle. Prove that S can also be* seen *from any other point inside the triangle ABC.*

7.1 Convex Hulls

A convex hull for the set of points a_1, a_2, ..., a_n is the smallest convex polygon which contains all n points, along its boundary or inside its interior. The convex hull of a set of points allows us to more formally define when an object is "inside" or "outside" an arbitrary set of points.

Example 7.1.1. *Prove that any line passing through the convex hull of a set of points must split the plane into two regions such that there exists at least one point of the set in both regions.*

Solution. Let S be the set of points, H be the convex hull of S, and ℓ be the line passing through H. Assume the contrary, that ℓ splits the plane (and H) into two regions such that one region contains all points of S. This implies that H is split into two convex polygons, of which one does not contain a single point of S. We can therefore remove this region of H to get a smaller convex polygon containing all points of S. However, the convex hull H is by definition the smallest such convex polygon containing all points of S, and since we have found a smaller such convex polygon we have reached a contradiction to our assumption. □

Exercise 7.1.2. (IMO Longlist 1966) *Given $n > 3$ points in the plane in general position. Prove that there exists a circle passing through (at least) 3 of the given points and not containing any other of the n points in its interior.*

It is also important to note that the vertices of the convex hull of a set S of points must be in S. These extremal points are often prime targets when examining problems with sets of points. Convex hulls can further be used in combination with the "sweeping lines" technique to aid in solving combinatorial geometry problems by making use of the more rigid "inside" and "outside" definitions these provide.

Exercise 7.1.3. *There are $2n$ points in the plane, no three of which are collinear. Show that there exists a line ℓ through 2 points which splits the plane in half with $n - 1$ points in each region.*

Exercise 7.1.4. *Show that for any set of n points in general position, these points are the vertices of a non-self-intersecting n-gon.*

8 Conclusion

Combinatorial geometry is a combination of combinatorics and geometry, and therefore both geometrical and combinatorial knowledge is needed to solve problems in these contexts. It is a broad area of mathematics which relates to everything from binomial coefficients, to the Pigeonhole principle, to convex hulls. In this text we have tried to touch on as many of these areas as possible, but have therefore foregone the opportunity to be more detailed and to go into more depth. (We encourage you to do this on your own, if you want). We have prioritised providing diverse problems, methods and techniques over specific theorems and facts. With this text it was our goal to make you better prepared for ANY combinatorial geometry problem you may encounter. But most of all, we hope that you found this text and topic interesting too!

Happy solving!

9 Additional Problems

Problems in combinatorial geometry often do not rely on one single method or principle to be solved, but require knowledge about multiple areas discussed in this text. Therefore there were some problems we felt were more suitable to be given to you outside the context of only one area. This is also an opportunity for you to encounter combinatorial geometry problems in a more natural problem solving setting.

Problem 9.1. *Given a set S of* $n + 3$ *points in general position in the plane such that any triangle with vertices in S has area at most A, show that there exists a triangle with area A with at least 2 vertices in S which covers* $\lfloor \frac{n}{4} \rfloor$ *points in S.*

Problem 9.2. (Continuation of *Exercise 5.5.*) *Show that in any set of points S in general position, there exist at least* $\lceil \frac{\binom{n}{4}}{n-4} \rceil$ *non-right angled triangles.*

Problem 9.3. ("Problems in Plane and Solid Geometry" by Prasolov) *Several points are marked on a circle, A is one of them. Which convex polygons with vertices in these points are more numerous: those that contain A or those that do not?*

Problem 9.4. ("Problems in Plane and Solid Geometry" by Prasolov) *Prove that the number of triangles with vertices in the vertices of a regular n-gon is equal to the integer nearest* $\frac{n^2}{12}$.

Problem 9.5. (Canadian IMO Training 2009) *A set S of points on a plane has the property that if* $A, B \in S$*, then the midpoint of A and B is also in S. Prove that either* $|S| = 1$ *or S is infinite.*

Problem 9.6. (Canadian IMO Training 2010) *A strip of width* w *is the set of all points which lie on or between two parallel lines that are a distance* w *apart. Let S be a set of* $n \geq 3$ *points on the plane such that any three different points of S can be covered by a strip of width 1. Prove that S can be covered by a strip of width 2.*

Problem 9.7. (Canadian IMO Training 2009) *Each point on the circumference of a circle is coloured either red or blue. Prove that there exist three distinct points on this circumference* X, Y, Z *all of the same colour such that* $|XY| = |XZ|$.

Problem 9.8. (SMT-Finals 2015) *Given is a finite amount of points in the plane and an equal amount of rays with the origin as their initial point. Is it always possible to pair up points and rays such that the translated rays beginning in their respective points do not intersect each other?*

Problem 9.9. (Blatic Way 2020) *Alice and Bob are playing hide and seek. Initially, Bob chooses a secret fixed point B in the unit square. Then Alice chooses a sequence of points* $P_0, P_1, \ldots, P_n$ *in the plane. After choosing* P_k *(but before choosing* P_{k+1}*) for* $k > 1$*, Bob tells "warmer" if* P_k *is closer to B than* P_{k-1}*, otherwise he says "colder". After Alice has chosen* P_n *and heard Bob's answer, Alice chooses a final point A. Alice wins if the distance AB is at most* $\frac{1}{2020}$*, otherwise Bob wins. Show that if* $n = 18$*, Alice cannot guarantee a win.*

Problem 9.10. (IMO 2011) *Let S be a finite set of at least 2 points in the plane. Assume that no three points of S are collinear. A* windmill *is a process that starts with a line* ℓ *going through a single point* $P \in S$*. The line rotates clockwise about the* pivot *P until the first time that the line meets some other point belonging to S. This point, Q, takes over as the new pivot, and the line now rotates clockwise around Q, until it next meets a point of S. This process continues indefinitely.*
Show that we can choose a point P in S and a line ℓ *going through P such that the resulting* windmill *uses each point of S as a* pivot *infinitely many times.*