The Method of Moving Points

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1 Introduction

The method of moving points is a powerful tool for solving geometry problems. It builds on the idea of showing that two *mappings*, i.e. functions between geometric objects, are the same. This can be done by showing that the mappings have a certain property and then checking three cases. If a geometric object (a point, line, circle etc.) is constructed such that criterion A holds and the objective of the problem is to prove that criterion B also holds for this object, then the method of moving points can be used by considering two mappings: one where criterion A holds and one where criterion B holds. Showing that the two mappings are equivalent will imply that criterion B holds and we will have a solution to our problem. This is a rough sketch of how the method works. After we have gone through the theory and looked at a few examples the details will hopefully be much clearer.

2 The Main Idea of the Method

Suppose you want to show that two real-valued functions f and g are the same. Just showing that $f(1) = g(1)$ and $f(2) = g(2)$ is not nearly enough, since f and g may be different at other points as figure 1 illustrates. However, if we could do some magic to show that f and g are linear, $f(1) = g(1), f(2) = g(2)$ would actually be enough to show that they are the same, since linear functions are uniquely determined by two points, see figure 2.

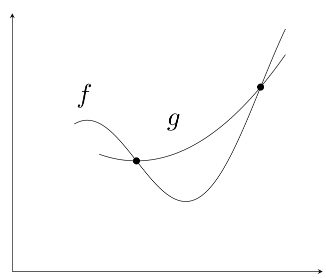


Figure 1: f and g are different.

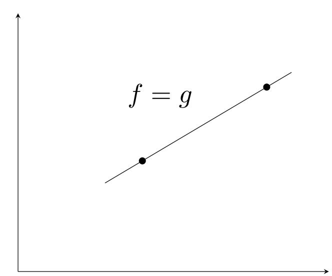


Figure 2: f and g are the same.

The idea behind the method of moving points is quite similar to this, but we instead look at functions between objects such as circles and lines. To prove that these functions are equivalent, i.e. equal for all inputs the functions are defined on, we prove that the functions are *projective maps* and that they are equal for three points [2].

3 The Cross-Ratio

In order to define what a projective map is, we need to introduce the so-called *cross-ratio*.

Definition 1. *Let A, B, C and D be four distinct points on a line. The cross-ratio, denoted $(A, B; C, D)$, is defined as follows:*

$$(A, B; C, D) = \frac{AC \cdot BD}{BC \cdot AD},$$

where AC, BD, BC and AD are distances and the orientation of the line determines the sign of each distance (i.e. $XY = -YX$ for two points X and Y).

The definition of the cross-ratio may seem arbitrary but as we will see it has some useful properties. The cross-ratio has its origin as a measure that is invariant under changes of perspective i.e. projective transformations. What this means in our case is that the cross-ratio is preserved by projection from a point. In figure 3 the cross-ratio is preserved during projection from line l to l' so that $(A, B; C, D) = (A', B'; C', D')$.

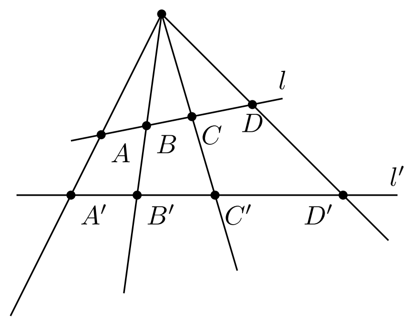


Figure 3: The cross-ratio is preserved during projection

Projection from a point is an example of a projective map. A more extensive list of projective maps will be given later. As a consequence of the cross-ratio being invariant under projection from a point, the cross-ratio can be defined not just for points on a line but also for lines in a *pencil of lines*, i.e. the set of all lines going through a fixed point. The cross-ratio of four intersecting lines can be defined as the cross-ratio of the four points of intersection with an arbitrary line not going through the intersection of the four lines.

Definition 2. *Let P be a pencil of lines. Given four lines $L_1, L_2, L_3, L_4 \in P$ and a line $l \notin P$, the cross-ratio is defined as*

$$(L_1, L_2; L_3, L_4) = (A, B; C, D),$$

where $A = L_1 \cap l$, $B = L_2 \cap l$, $C = L_3 \cap l$ and $D = L_4 \cap l$. Through trigonometry, it can be shown that this is equivalent to

$$(L_1, L_2; L_3, L_4) = \pm \frac{\sin \angle(L_1, L_3) \sin \angle(L_2, L_4)}{\sin \angle(L_2, L_3) \sin \angle(L_1, L_4)}$$

where $\angle(L_i, L_j)$ denotes the angle[1] *between lines L_i and L_j and the sign is positive if and only if one of the angles formed by L_1 and L_2 does not contain L_3 or L_4 [1].*

Furthermore, the cross-ratio can be defined for points on a circle.[2]

Definition 3. *The cross ratio for four points P_1, P_2, P_3, P_4 on a circle is given by*

$$(P_1, P_2; P_3, P_4) = (P_1Q, P_2Q; P_3Q, P_4Q).$$

[1] Two lines actually form angles with both sizes θ and $180° - \theta$, but since $\sin(\theta) = \sin(180° - \theta)$ this is not a problem for us.

[2] More generally for points on a conic section, but only circles will be discussed here.

where Q is any[3] *point on the circle.*

Because of the inscribed angle theorem, we know that the value of $\sin\angle P_iQP_j$ is independent of Q. From definition 2 we can then see that the value of $(P_1Q, P_2Q; P_3Q, P_4Q)$ is the same for all choices of Q on the circle and so the cross ratio for circles is well defined.

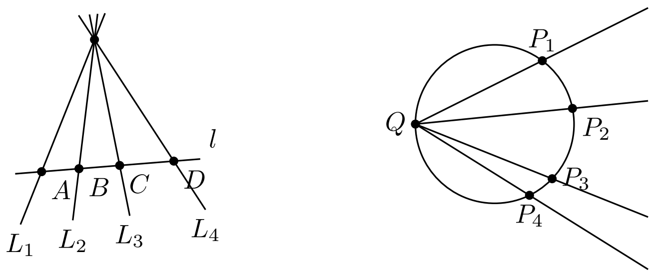


Figure 4: The cross-ratio for lines in a pencil of lines and points on a circle.

4 Projective Maps

We are now ready to define what a projective map is:

Definition 4. *Let $\mathcal{C}_1$, $\mathcal{C}_2$ be two objects for which the cross ratio is defined. A function $f : \mathcal{C}_1 \to \mathcal{C}_2$ is called a projective map if it preserves the cross-ratio, that is if*

$$(A, B; C, D) = (f(A), f(B); f(C), f(D))$$

where A, B, C, $D \in \mathcal{C}_1$.

Why would any of this be useful? Recall that a linear function is uniquely determined by 2 points. Similarly, a projective map is uniquely determined by 3 points. We have the following very important theorem:

Theorem 1. *Two projective maps $f : \mathcal{C}_1 \to \mathcal{C}_2$ and $g : \mathcal{C}_1 \to \mathcal{C}_2$ are equivalent if $f = g$ for three distinct points on $\mathcal{C}_1$.*

Proof. Let A_1, A_2 and A_3 be distinct points on $\mathcal{C}_1$ and B_1, B_2 and B_3 be points on $\mathcal{C}_2$ such that $f(A_i) = g(A_i) = B_i$ for $i \in \{1, 2, 3\}$. Note that B_1, B_2 and B_3 are also distinct.[4] Now consider a point $X \in \mathcal{C}_1 \setminus \{A_1, A_2A_3\}$. Since f and g are both projective, $(A_1, A_2; A_3, X) = (B_1, B_2; B_3, f(X)) = (B_1, B_2; B_3, g(X))$. From the definition of the cross-ratio, we get that

$$\frac{B_1B_3 \cdot B_2f(X)}{B_2B_3 \cdot B_1f(X)} = \frac{B_1B_3 \cdot B_2g(X)}{B_2B_3 \cdot B_1g(X)} \iff \frac{B_2f(X)}{B_1f(X)} = \frac{B_2g(X)}{B_1g(X)}.$$

Since $B_2f(X) = B_1f(X) - B_1B_2$ and $B_2g(X) = B_1g(X) - B_1B_2$, this is again equivalent to

$$\frac{B_1f(X) - B_1B_2}{B_1f(X)} = \frac{B_1g(X) - B_1B_2}{B_1g(X)} \iff 1 - \frac{B_1B_2}{B_1f(X)} = 1 - \frac{B_1B_2}{B_1g(X)} \iff B_1f(X) = B_1g(X).$$

This implies that $f(X) = g(X)$. □

[3] If $Q = P_i$, the line P_iQ is taken to be tangent the the circle at that point.

[4] Since A_1, A_2 and A_3 are distinct, the cross-ratio is neither 0 nor involves division by zero. This is still the case after the projective map has been applied, and thus B_1, B_2 and B_3 are also distinct.

This theorem is the central result that the method of moving points relies on. It is a very powerful tool since we can prove that two projective maps are the same by checking that they are equal for just three cases.

The other part of the method is to prove that functions between geometric objects are projective maps. The definition of projective maps says that the cross-ratio being invariant is sufficient for a function to be projective, but proving that a map is projective using only the definition can be complicated. Instead, it often serves us to decompose a function into a composition of maps that are known to be projective. This proves that the map is projective because of to the following very important fact:

Theorem 2. *The composition $f \circ g$ of two projective maps f and g, is projective.*

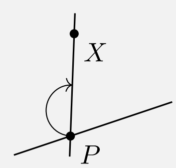
Proof. The proof is left as an exercise to the reader. ☐

5 A List of Projective Maps

Projection from a point has already been mentioned as an example of a projective map, but it is far from the only one. Here is a short list of the most important types of projective maps:

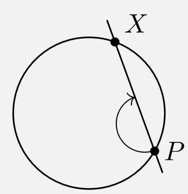
Line Projection

Given a fixed line and point X not on the line, the map taking every point P on the line to a line XP, or vice versa, is projective.



Circle Projection

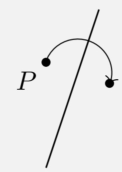
Given a fixed circle and point X on it, the map taking every point P on the circle to a line XP, or vice versa, is projective.



When considering line and circle projections taking a point to a line through X and then taking that line to a point we will abbreviate the description and call it a projection from X between a line/circle and a line/circle. However if you want to apply an intermediate map to the line it will be useful to think of the maps as taking points to lines and vice versa.

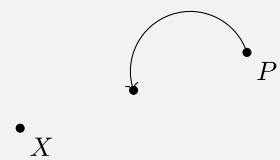
Line Reflection

Given a fixed line, the map reflecting every point P on some line or circle through this line is projective.



Point Scaling

Given a fixed point X, the map scaling with centre X every point P on some line or circle by a fixed value is projective. Two special cases are the map taking P to the midpoint of segment XP, and the map reflecting P through X.



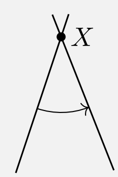
Definition 4 says that we need to prove that the cross-ratio is invariant to show that point scaling is projective. Point scaling simply scales all distances by a fixed value α. The distances AC, BD, BC and AD, then become $A'C' = \alpha \cdot AC$, $B'D' = \alpha \cdot BD$, $B'C' = \alpha \cdot BC$ and $A'D' = \alpha \cdot AD$ respectively. We see that the cross-ratio $(A', B'; C', D')$ is equal to $(A, B; C, D)$ since

$$(A', B'; C', D') = \frac{A'C' \cdot B'D'}{B'C' \cdot A'D'} = \frac{\alpha \cdot AC \cdot \alpha \cdot BD}{\alpha \cdot BC \cdot \alpha \cdot AD} = \frac{AC \cdot BD}{BC \cdot AD} = (A, B; C, D),$$

which proves that point scaling is projective.

Rotation

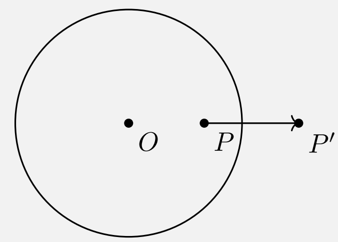
Given a fixed point X, the map rotating every line through X by some fixed value is projective.



Under rotation the cross-ratio of lines is invariant because the cross-ratio only depends on the angles between the lines by the seconds part of definition 2. Since every line through X is rotated by the same fixed value, the angles between those lines are the same and thus the cross-ratio is preserved. This proves that the rotation of every line through a point X by a fixed angle is projective by definition 4.

Circle Inversion

Given a point O and a radius r the map taking any point P on a line or circle to the point P' on the ray OP such that $OP \cdot OP' = r^2$ is projective.



6 Interlude: Basics of Circle Inversion

To those unfamiliar with inversion a bit more explaining will be necessary to fully grasp it. Here we will lay out the basic properties of inversion without fully explaining every detail for the sake of brevity. However, we encourage the reader to look more into inversion since it is an amazing subject in its own right.

The reason we are interested in inversion is because it preserves the cross-ratio, meaning that it is projective, and it allows us to better work with circles by transforming them into lines. The formula $OP \cdot OP' = r^2$ tells us where points go but we want to understand what happens to sets of points such as circles or lines since the method of moving points entails moving along such sets.

The image of a circle not going through the centre O of the inversion, is a circle. Since a circle is defined by three points, the way you would construct the image in practice is to take the inverse of three points and draw the inverted circle through them. Furthermore, if the circle goes through the centre of inversion O it is mapped to a line not going through O. Note that inversion is its own inverse, i.e., applying inversion twice with the same centre and radius gives back the original picture. This tells us that a line not going through O is mapped onto a circle going through O. A line going through O is mapped onto itself which should be clear from the definition.

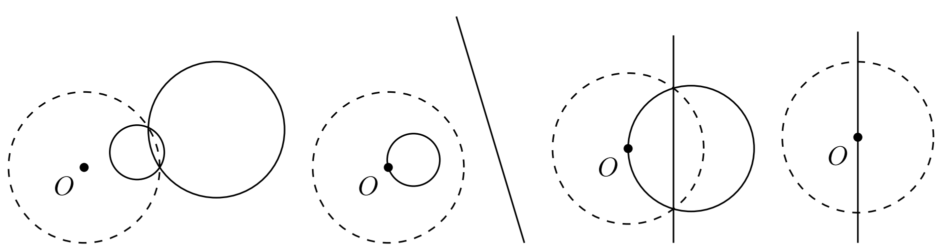


Figure 5: Inversion maps circles to circles unless the circle goes through the centre of inversion O, then the circle maps to a line not going through O and vice versa. A line through O maps to itself.

7 The Method of Moving Points

Now that we are more familiar with projective maps, we can present a protocol for using the method of moving points in geometric problems.

The Method of Moving Points

1. Define two functions f and g that take some point P on a line or circle to X and Y respectively. These functions should correspond to two different constructions: one given by the problem statement and one given by what ought to be proven.

2. Show that both f and g are projective maps, e.g. by showing that they are compositions of projective maps.

3. Find three positions of P for which X and Y coincide. By Theorem 1, this implies that f and g are equivalent and the proof is complete.

Why is this called the method of *moving points*? Because we can say that we "move" the point P along some curve which in turn moves the points X and Y.

8 Points at infinity

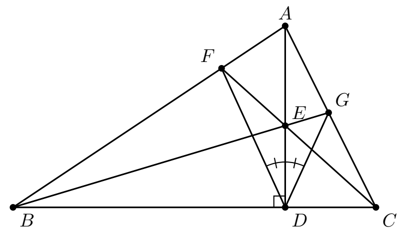
Since we need three points to choose for P it would be useful to have as many points available as possible. We therefore introduce the concept of *points at infinity* and the *projective plane.* The projective plane can be seen as an extension of the Euclidean plane. In the Euclidean plane every two points define a line (the line through the points) and every two lines define a point (the intersection) except when the lines are parallel. To resolve this asymmetry, points at infinity, defined by the intersection of parallel lines, are added to the Euclidean plane to form the projective plane. A point at infinity can be defined by a direction (the direction the parallel lines are facing) and together all points at infinity form a *line at infinity.* Let us for example say that we have a line l, a point A not on the line l and a point P moving along the line l. By choosing P to be the intersection of l and the line at infinity the line AP will be parallel to l which might make it easy to show that two maps involving this construction are equal. This works since the cross-ratio can be defined for points at infinity. Let us say that we want to compute $(A,B;C,D)$ and that D is a point at infinity. We then drop the distances involving D from the expression giving us

$$(A,B;C,D) = \frac{AC}{BC}.$$

The thought behind why we do this is that the two infinite distances BD and AD cancel each other. Points at infinity also let us define what happens to the centre of a circle inversion. We say that if a line or circle goes through the centre of inversion, this centre is mapped to the point at infinity on the inverted line. In the same way the point at infinity on a line is mapped to the centre of inversion.

9 Example Problems

Example 1. Given a triangle ABC and some point E on the altitude through A. Let D be the foot of the altitude through A, let F be the intersection of AB and CE and let G be the intersection of AC and BE. Prove that $\angle ADF = \angle ADG$.



Solution. First we choose to move the point E. The strategy to solve this problem is to prove that the unique point $X \in AC$ which lies on the line BE is the very same point as the unique point $Y \in AC$ defined such that $\angle ADF = \angle ADY$. If X and Y coincide, G is positioned such that the angles in question are equal and the proof is complete.

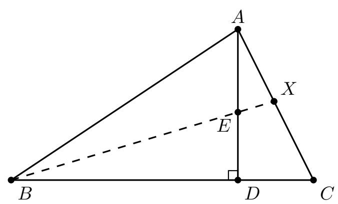


Figure 6: $E \mapsto X$

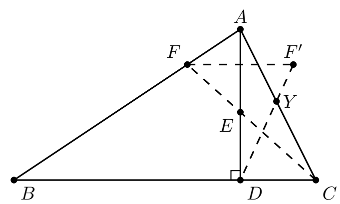


Figure 7: $E \mapsto Y$

Consider the map $f : AD \to AC$ sending the point $E \in AD$ to $X \in AC$ such that $X = AC \cap BE$. This map is a projection from a line to another line and is thus projective. Furthermore, the map $g : AD \to AC$, which sends $E \in AD$ to $Y \in AC$ such that $\angle ADF = \angle ADY$ is projective since it is a composition of projective maps. The map can be broken up into the steps $E \mapsto F \mapsto F' \mapsto Y$ where F' is the reflection of F through AD. The maps $E \mapsto F$ and $F' \mapsto Y$ are projections and $F \mapsto F'$ is a projective map since it is a reflection.

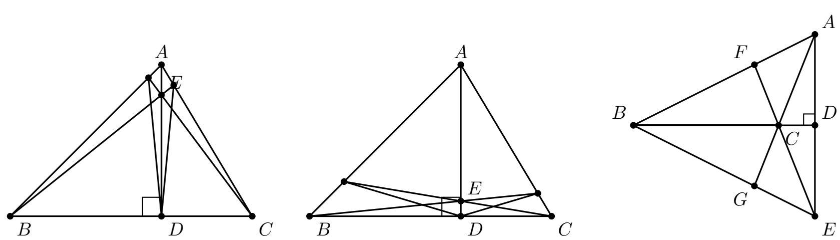


Figure 8: Let E be A, D and the reflection of A over BC.

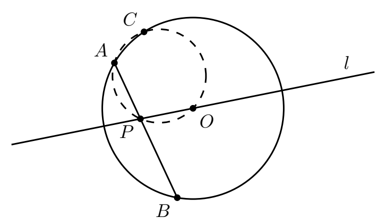
Using the method of moving points, all that is left is to check that f and g are equal for three different choices of E.

- Firstly consider when E lies on A. Then f maps E to A and g trivially maps E to A and both angles are zero.
- Secondly, if E lies on D, f maps E to C just like g since both angles are right angles.

- Finally, consider when E lies on the reflection of A over line BC. By symmetry, $\angle ADX = \angle EDF$ and since $\angle EDF = \angle EDF'$ we have $\angle ADX = \angle EDF'$. Because A, D and E lie on a line with D between A and E, this implies that X, D and F' lie on a line and so the projection of F' on AC through D is X. Thus $X = Y$ and our final case is complete.

■

Example 2. (SMT qualification round 2018) Let AB be a chord in a circle with centre O. The line l goes through O and intersects the chord AB in the point P. Let C be the reflection of the point B through the line l. Show that the points A, C, O and P lie on a circle.



Solution. Let Γ denote the circle with centre O. We first see that in the special case when B lies on l, B coincides with C and P. Thus C and P are the same point and it is trivial that the points lie on a circle. To handle the general case we choose to move A on the circle Γ and fix B and l. Consider the map $f : \Gamma \to l$ defined by mapping A to $X = l \cap AB$. This map is a projection from a circle to a line through a point on the circle which clearly is projective. Furthermore, the map $g : \Gamma \to l$ defined to map A to $Y = l \cap (AOC)$ i.e. the not already known intersection of the line l and the circumcircle to AOC, is projective. To see why this map is projective we apply inversion with respect to a circle centred at C. Let $\phi(t)$ denote the image of t under this inversion. The map g can be decomposed into $A \mapsto \phi(A) \mapsto \phi(Y) \mapsto Y$. Here $\phi(A) \mapsto \phi(Y)$ denotes the projection of $\phi(A)$ to the the circle $\phi(l)$ from the point $\phi(O)$. Thus the map g is projective.

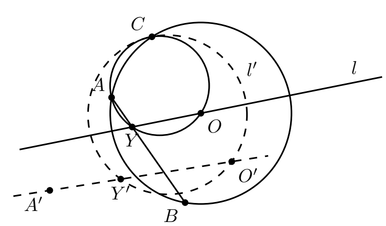


Figure 9: The image of l under inversion with centre C is a circle l' and the image of (AOC) is a line.

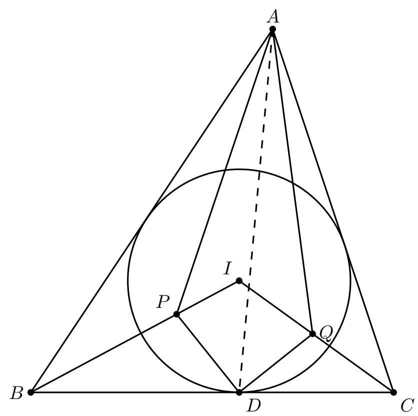
To prove that f is equivalent to g, which would imply that P lies on the unique circle going through the points A, C and O, we check three cases for A.

- Let A be the two intersections of Γ and l. Then f maps A to itself. Since A is the other point (AOC) intersects the line l, g also maps A to itself. This gives us that the maps are equal for two points.

- Let A be the intersection of the line BO and Γ not equal to B. Then f maps A to O and by symmetry (AOC) is tangent to l at O so g maps A to O too.

■

Example 3. (Serbia MO 2018) Let $\triangle ABC$ be a triangle with incentre I. Points P and Q are chosen on segments BI and CI such that $2\angle PAQ = \angle BAC$. If D is the touch point of incircle and side BC, prove that $\angle PDQ = 90°$.



Solution. We move P. Consider a mapping $f : BI \to CI$ sending $P \in BI$ to $X \in CI$ such that $\angle PAX = \frac{1}{2}\angle BAC$ and another mapping $g : BI \to CI$ sending $P \in BI$ to $Y \in CI$ such that $\angle PDY = 90°$. The map f is projective since $P \mapsto AP \mapsto AX \mapsto X$ is composition of projective maps. $P \mapsto AP$ is a map from a point to a pencil of lines through A and preserves the cross ratio. Furthermore, $AP \mapsto AX$ is a rotation by the fixed angle $\frac{1}{2}\angle BAC$ and thus projective so f is projective. Similarly g is projective. If we show that both maps are equivalent it will follow that P is sent to Q by both maps and thus $\angle PDQ = 90°$. We show they are equivalent by showing that the maps are equal for three positions of P.

- Let $P = B$. Since $\angle BAI = \frac{1}{2}\angle BAC$ f maps P to I. The map g maps B to the same point since BD is tangent to the incircle and the radius DI is perpendicular to the tangent.
- Let $P = I$. With a smiliar argument as before both f and g map P to C.
- Let P, Q be the centres of the inscribed circles of $\triangle ABD$, $\triangle ACD$ respectively. Since $\angle PAQ = \angle PAD + \angle DAQ = \frac{1}{2}\angle BAD + \frac{1}{2}\angle DAC = \frac{1}{2}\angle BAC$ f maps P to Q and since $\angle PDQ = \angle PDA + \angle ADQ = \frac{1}{2}\angle BDA + \frac{1}{2}\angle ADC = \frac{1}{2}\angle BDC = 90°$, g also maps P to Q.

■

10 Conclusion

We have looked at examples dealing with projection, reflection, inversion and rotation so we hope you have a good understanding of the protocol now. Furthermore, we hope that this topic has been both insightful and entertaining. In this paper we covered far from everything there is to know about the discussed topics and further study is encouraged. For example, the article *The Method of Animation* by Zack Chroman, Gopal K. Goel and Anant Mudgal provides a powerful generalisation of the method. Nonetheless, since mathematics is not a spectator sport we have compiled a list of exercises so you may apply these ideas yourself and develop new problem solving skills.

11 Exercises for the Reader

Problem 1. Let AB be a diameter of circle ω. l is the tangent line to ω at B. Take two points C, D on l such that B is between C and D. E, F are the intersections of ω and AC, AD, respectively, and G, H are the intersections of ω and CF, DE, respectively. Prove that $AH = AG$.

Problem 2. Let ABC be a triangle with circumcircle (O). The tangent to (O) at A intersects the line BC at P. E is an arbitrary point on the line PO, and $D \in BE$ is such that $AD \perp AB$. Prove that $\angle EAB = \angle ACD$.

Problem 3. Let Γ be a circle, O its centre and l a line. The perpendicular through O to l intersects Γ at A and B. Let P, Q be two points on Γ, and $PA \cap l = X_1$, $PB \cap l = X_2$, $QA \cap l = Y_1$, $QB \cap l = Y_2$. Prove that the circumcircles of $\triangle AX_1Y_1$ and $\triangle AX_2Y_2$ intersect on Γ.

Problem 4. In $\triangle ABC$ $\angle B$ is obtuse and $AB \neq BC$. Let O be the circumcentre and ω be the circumcircle of this triangle. N is the midpoint of arc ABC. The circumcircle of $\triangle BON$ intersects AC on points X and Y. Let $BX \cap \omega = P \neq B$ and $BY \cap \omega = Q \neq B$. Prove that P, Q and the reflection of N with respect to line AC are collinear.

Problem 5. (USA Winter TST for IMO 2019) Let ABC be a triangle and let M and N denote the midpoints of $\overline{AB}$ and $\overline{AC}$, respectively. Let X be a point such that $\overline{AX}$ is tangent to the circumcircle of triangle ABC. Denote by ω_B the circle through M and B tangent to $\overline{MX}$, and by ω_C the circle through N and C tangent to $\overline{NX}$. Show that ω_B and ω_C intersect on line BC.

References

[1] Evan Chen. *Euclidean geometry in mathematical olympiads*. Vol. 27. American Mathematical Soc., 2021.

[2] Vladyslav Zveryk. “The Method of Moving Points”. In: (July 2019).