The Orthocentre and the Pedal Triangle of a Triangle

Axel Hagerud, Rilind Hoti, Neo Dahlfors & Isak Fleig*
Norra real gymnasieskola

May 2021

Contents

*Minervagymnasium

1 Introduction

In geometry one of the most common objects is the triangle. Being able to make further constructions from the information given is the key to solving all but the most basic problems you may encounter in geometry.

This document is about the orthocentre and the pedal triangle of a triangle, which will be introduced shortly.

The orthocentre is one of the most important points of a triangle. The pedal triangle, especially the orthic pedal triangle, is also frequently key to solving problems.

2 The location of the orthocentre

To find the orthocentre of a triangle ABC, let R, S, T be the orthogonal projections of the vertices A, B, C onto the lines generated by $\overline{BC}, \overline{CA}, \overline{AB}$ respectively. The orthocentre H is then the point where the lines $\overline{AR}$, $\overline{BS}$, $\overline{CT}$ intersect.

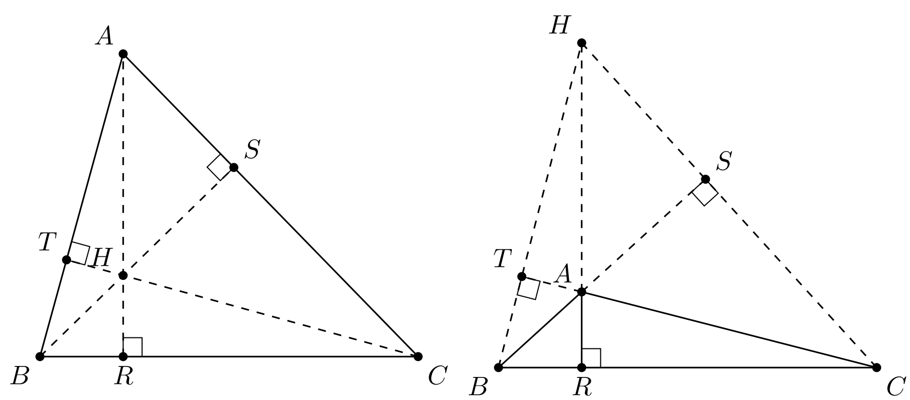

Figure 1: H is the orthocentre of triangle ABC.

Exercise 2.1. *Investigare when the orthocentre lies inside, and when it lies outside the triangle. Does the orthocentre ever lie on the perimeter of the triangle?*

The very existence of an orthocentre relies on the fact that $\overline{AR}$, $\overline{BS}$, $\overline{CT}$ always intersect at a single point.

Theorem 2.1. *The altitudes of a triangle meet at a point, the orthocentre.*

Proof. To prove this, the fact that the perpendicular bisectors of a triangle meet at a point will be used. This can be shown as follows:

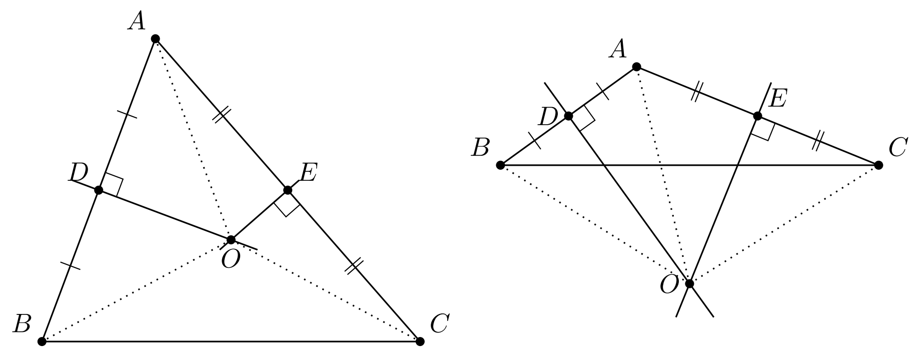

Figure 2: The perpendicular bisectors of a triangle meet at one point, the centre of the circumscribed circle. Here an acute and an obtuse triangle are shown.

Lemma (The perpendicular bisectors of a triangle intersect).
Consider the triangle ABC. The perpendicular bisectors of the sides AB and AC meet at a point O and intersect $\overline{AB}$ and $\overline{AC}$ at D and E respectively. Since $|AD| = |BD|$ and $\angle ADO = 90° = \angle BDO$, the triangles ADO and BDO are congruent by side-angle-side. This implies that $|AO| = |BO|$. Similarly, it can be shown that $|AO| = |CO|$, implying that $|BO| = |CO|$. Using simple trigonometry, this implies that O lies on the perpendicular bisector of side BC, meaning that the perpendicular bisectors of a triangle meet at one point. □

Given a triangle ABC, three additional triangles, congruent with ABC, can be constructed as in Figure 3:

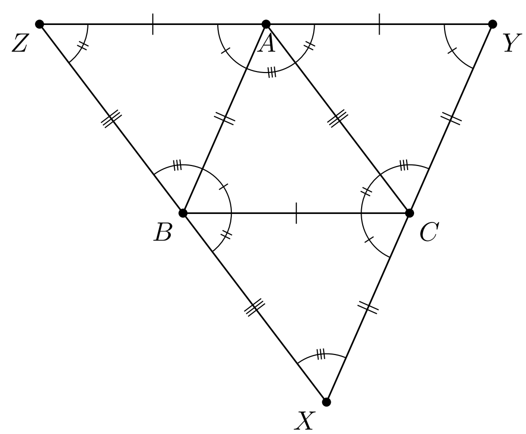

Figure 3: Triangles $\triangle ABZ, \triangle ACY$ and $\triangle BCX$ congruent with triangle $\triangle ABC$ are constructed on the sides of $\triangle ABC$.

From the way the triangles are constructed it immediately follows that $\overline{BC}$ is parallel with $\overline{YZ}$, which can be noticed by the angles in Figure 3. Since the altitude of triangle ABC through A is perpendicular to BC, it must also be perpendicular to $\overline{YZ}$. This, in addition to the fact that $|AY| = |AZ|$ means that said altitude also is the perpendicular bisector of segment $\overline{YZ}$. Using identical arguments, it can be shown that all the altitudes of triangle ABC are the perpendicular bisectors of triangle XYZ. Since the perpendicular bisectors of a triangle are concurrent, it follows that the altitudes of any given triangle ABC always meet at one point. □

The proof above works without need for modification for acute and obtuse triangles. The same result can also be achieved using Ceva's Theorem or the Euler Line.

3 The pedal and orthic triangles

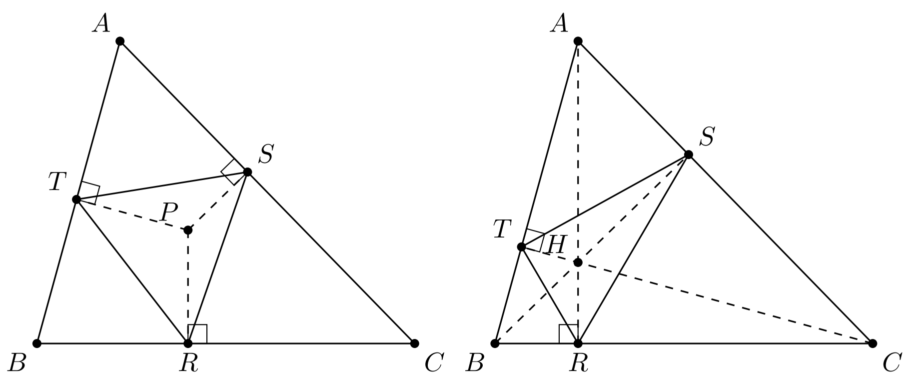

Figure 4: There are many pedal triangles STR of the triangle ABC. The picture to the right shows a pedal triangle that is also the orthic triangle.

A pedal triangle can be obtained by projecting a point onto the sides of a triangle and then connecting the projections, as in Figure 4 where $\triangle SRT$ is a pedal triangle of $\triangle ABC$. Depending on the chosen point for projection, a pedal triangle will have different properties. Both the medial triangle and the intouch triangle are pedal triangles, generated by projecting the centre of the circumscribed or inscribed circle of a triangle onto its sides.

If the point being projected is the orthocentre, the pedal triangle becomes the orthic triangle. The orthic triangle can also be obtained by connecting the three points where the altitudes intersect with the lines generated by the sides of the triangle. The orthic triangle has some very interesting properties.

Theorem 3.1. *The orthocentre of the acute angled triangle $\triangle ABC$ is also the centre of the inscribed circle of the triangle $\triangle STR$.*

Proof. The centre of the inscribed circle is the point where the internal angle bisectors meet. Thus, proving the theorem is equivalent to showing that the angles $\angle RST$, $\angle RTS$ and $\angle SRT$ have the respective angular bisectors $\overline{SH}$, $\overline{TH}$ and $\overline{RH}$. Due to rotational symmetrical reasons this only needs to be proven for one of the three vertices mentioned above.

To show that $\angle TRH = \angle SRH$ first construct a circle with the side AB as its diameter. The inverse of the inscribed angle theorem implies that the points R and S also lie on the arc of this circle. The inscribed angle theorem is now used to show that $\angle ABS = \angle ARS$. By constructing an additional circle with diameter AC the same reasoning can be used to show that $\angle ART = \angle ACT$.

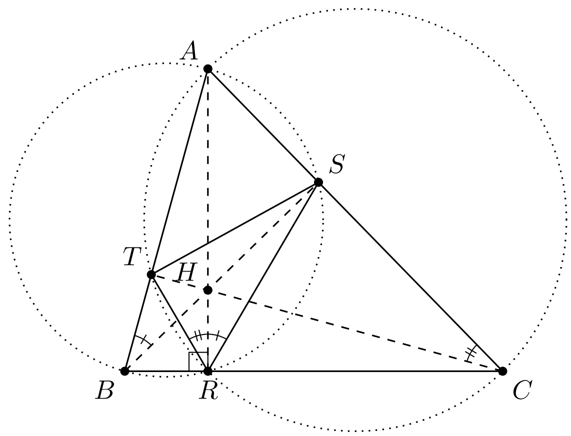

Figure 5: The inscribed angle theorem is used to show that $\angle ABS = \angle ARS$ and $\angle ART = \angle ACT$.

The angles $\angle BTH$ and $\angle CSH$ are equal, both are $90°$. Since $\angle BHT$ and $\angle SHC$ are opposite angles this means that the triangles $\triangle BHT$ and $\triangle CHS$ share two angles. This implies that the triangles are similar, which means that $\triangle TBH = \angle SCH$.

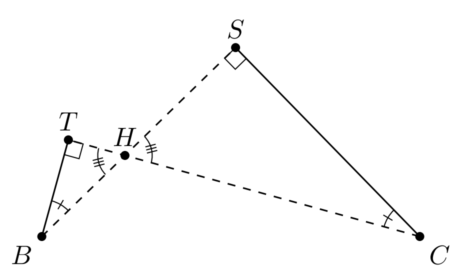

Figure 6: The triangles are similar. That implies that $\angle TBH = \angle SCH$.

Returning to the configuration in Figure 5 the above shown statement now means that $\angle TRH = \angle SRH$. This proves that $\overline{RH}$ is the bisector of $\angle TRS$. This implies that H is the centre of the inscribed circle of the orthic triangle, since the bisectors of the three angles $\overline{SH}$, $\overline{TH}$ and $\overline{RH}$ all meet in H, the orthocentre of $\triangle ABC$. The proof is thus complete. □

Exercise 3.1. *In Figure 5, show that the triangles ABC and AST are similar.*

Exercise 3.2. *It is well known that the area of a triangle is $\frac{base \cdot height}{2}$. Show that this formula gives the same area irregardless of the choice of base of the triangle.*

4 Inscribed quadrilaterals associated with the orthocentre

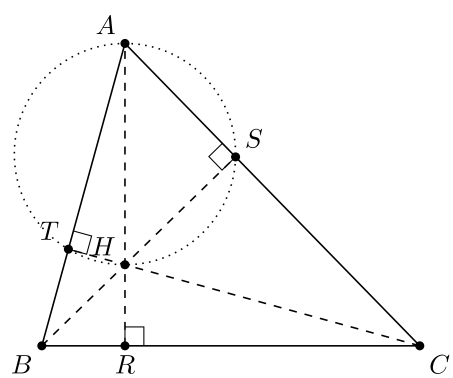

Figure 7: H is the orthocentre and $\triangle RST$ is the orthic triangle of $\triangle ABC$.

When studying the configuration of the orthocentre H of a triangle ABC along with the vertices S, R and T of its orthic triangle as in Figure 7, one can observe a strong correlation between these points and inscribed quadrilaterals, which can be very useful for problem solving.

In Figure 7, the quadrilateral $ATHS$ is inscribed in a circle. This is because a quadrilateral is inscribed in a circle if and only if two opposite angles add up to 180°, and

$$\angle ATH + HSA = 90^\circ + 90^\circ = 180^\circ.$$

The quadrilateral $BCST$ is also inscribed in a circle by the inverse of the inscribed angle theorem, since

$$\angle BTC = \angle BSC = 90^\circ.$$

Similarly, it can be shown that all of the quadrilaterals

$$ATHS, BRHT, CSHR, BCST, CATR \text{ and } ABRS$$

are inscribed in circles, and since both $90° = 90°$ and $90° + 90° = 180°$, the circles will remain even if different configurations permute the order of the points. An example will illustrate how these inscribed quadrilaterals may be used to solve problems.

Example 4.1 (IMO Shortlist 2010 G1). *Let ABC be an acute triangle with D, E, F the feet of the altitudes lying on $\overline{BC}, \overline{CA}, \overline{AB}$ respectively. One of the intersection points of the line $\overline{EF}$ and the circumcircle is P. The lines $\overline{BP}$ and $\overline{DF}$ meet at point Q. Prove that $|AP| = |AQ|$.*

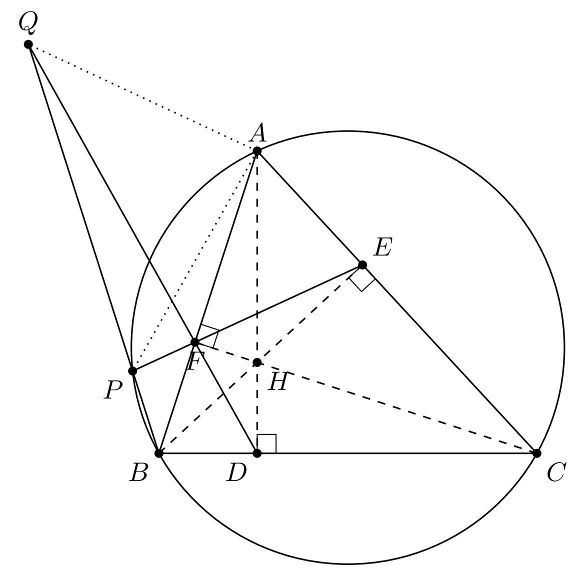

Figure 8: The construction given in the problem, with the added orthocentre H of $\triangle ABC$.

The following solution holds only for the configuration given in Figure 8. If the other intersection of $\overline{EF}$ and the circumcirle of $\triangle ABC$ is chosen as P, the proof is similar but minor changes are needed in the angle calculations.

Solution. Note that

$$\begin{aligned}
\angle QPA &= 180° - \angle BPA && \text{(Supplementary angles)} \\
&= \angle BCA && \text{(Inscribed quadrilateral } APBC\text{)} \\
&= \angle DCA && (D \text{ on ray } \overrightarrow{CB}) \\
&= 180° - \angle DFA && \text{(Inscribed quadrilateral } AFDC\text{)} \\
&= \angle QFA && \text{(Supplementary angles)}
\end{aligned}$$

which implies that the quadrilateral $AQPF$ is inscribed in a circle because of the inverse of the inscribed angle theorem (P, F lie on the same side of AQ).

But then

$$
\begin{aligned}
\angle PQA &= 180^\circ - \angle PFA && \text{(Inscribed quadrilateral } AQPF\text{)} \\
&= 180^\circ - \angle BFE && \text{(Vertical angles)} \\
&= \angle BCE && \text{(Inscribed quadrilateral } BFEC\text{)} \\
&= \angle BCA && (A \text{ on ray } \overrightarrow{CE}) \\
&= 180^\circ - \angle BPA && \text{(Inscribed quadrilateral } APBC\text{)} \\
&= \angle QPA && \text{(Supplementary angles)}
\end{aligned}
$$

gives that $\angle PQA = \angle QPA$. This means that $\triangle APQ$ is isosceles and it follows that segments AP and AQ are of equal length. □

5 The reflections of the orthocentre

One of the more unexpected, but surprisingly useful properties of the orthocentre, is it's symmetry with respect to the sides of the triangle. As a quick reminder, the reflection of a point P over a line ℓ is the point P' such that PP' is perpendicular to ℓ and the distances from both points to ℓ are equal.

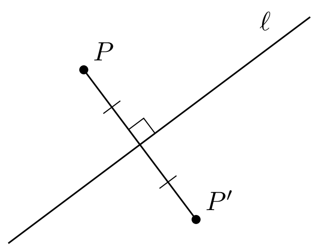

Figure 9: P' is the reflection of P over the line ℓ.

Theorem 5.1. *In the triangle ABC with orthocentre H, the reflections of H over the lines $\overline{AB}, \overline{BC}$ and $\overline{CA}$ lie on the circumscribed circle of the triangle ABC.*

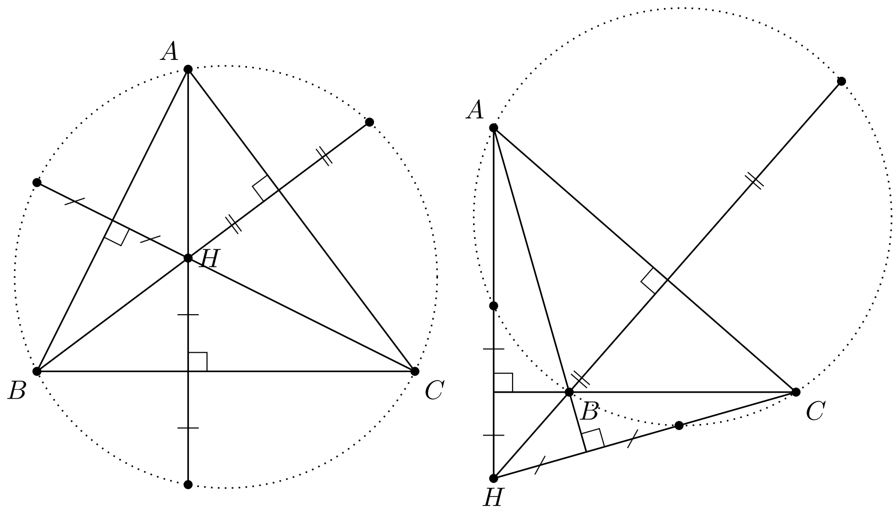

Figure 10: The reflections of the orthocentre over the sides of the triangle.

The following proof assumes that the triangle is acute, as in the leftmost picture of Figure 10. The proof of the obtuse case can be obtained by minor changes in the angle calculations.

Proof. Consider only the reflection of H over $\overline{BC}$ since the two other calculations will be identical in approach. Also, let D and E be the feet of the perpendiculars from A and B to $\overline{BC}$ and $\overline{CA}$ respectively, and let H' be the reflection of H over $\overline{BC}$.

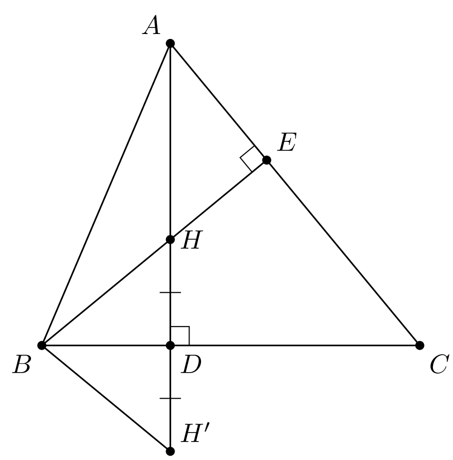

Figure 11: H' is the reflection of H over BC.

By definitions, $\angle HDB = 90^\circ = \angle BDH'$ and segments HD and DH' are equal in length. Triangles HBD and $H'BD$ are then congruent by side-angle-side, allowing for the following calculation:

$$\begin{aligned}
\angle H'BC &= \angle H'BD && (D \text{ on ray } \overrightarrow{BC}) \\
&= \angle DBH && (\triangle H'BD \cong \triangle HBD) \\
&= 90^\circ - \angle BHD && (\text{Sum of angles in } \triangle HBD) \\
&= 90^\circ - \angle EHA && (\text{Vertical angles}) \\
&= 90^\circ - (90^\circ - \angle HAE) = \angle HAE && (\text{Sum of angles in } \triangle EHA) \\
&= \angle H'AC && (H', C \text{ on rays } \overrightarrow{AH}, \overrightarrow{AE}).
\end{aligned}$$

Thus $\angle H'BC = \angle H'AC$, and since B and A lie on the same side of $\overline{H'C}$, the inverse of the inscribed angle theorem gives that A, B, H' and C all lie on a circle. Since this circle passes through both A, B and C, it is the circumscribed circle of triangle ABC which concludes the proof. □

Exercise 5.1. *It is possible to reflect objects over points as well as lines. A point P' is said to be the reflection of P over the point Q if Q is the midpoint of segment PP'. Show that the reflections of the orthocentre over the midpoints of a triangle also lie on the circumscribed circle of the triangle.*

Example 5.1. (Polish MO Finals 2019) *Let ABC be an acute triangle. The points X and Y lie on the segments AB and AC, respectively, and are such that $|AX| = |AY|$ and the segment XY passes through the orthocentre of the triangle ABC. The lines tangent to the circumcircle of the triangle AXY at the points X and Y intersect at the point P. Prove that the points A, B, C, P are concyclic.*

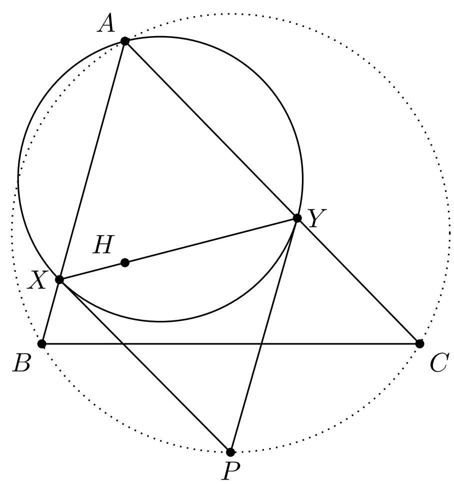

Figure 12: The construction given in the problem statement.

Before contemplating the solution, the reader should be aware of the following result.

Theorem 5.2 (Tangent-chord theorem)**.** *Let A, B and C be points on a circle, and let P be a point on the tangent to the circle at A such that B and P are on opposite sides of $\overline{AC}$. Then $\angle ABC = \angle PAC$ as in Figure 13.*

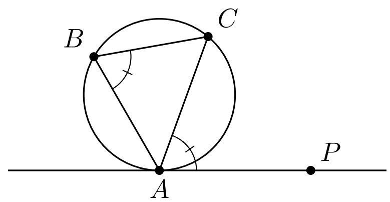

Figure 13: The tangent-chord theorem.

It is a very useful tool in handling tangency conditions, but sadly often goes overlooked in the Swedish mathematics curriculum.

Solution. Let R and Q be the intersections closest to A of $\overline{PX}$ and $\overline{PY}$ with the circumscribed circle of triangle ABC.

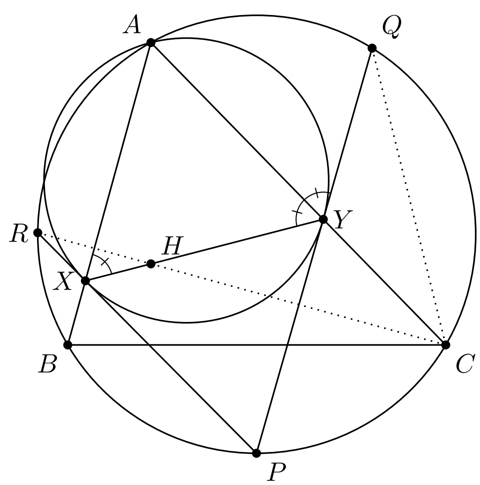

Figure 14: R and Q are defined in the section above.

Close inspection of Figure 14 suggests that Q and R are reflections of H over the lines AC and AB. This is indeed the case since the calculation

$$\begin{aligned} \angle HYA &= XYA && (X \text{ on ray } \overrightarrow{YX}) \\ &= \angle YXA && (\triangle AXY \text{ isosceles}) \\ &= \angle QYA && (\text{Tangent-chord theorem}) \end{aligned}$$

gives that $\angle HYA = \angle QYA$, implying that the reflection of H over $\overline{CA}$ lies on ray $\overrightarrow{YQ}$ due to symmetry. The fact that the reflection of H over $\overline{CA}$ lies on the circumscribed circle of triangle ABC now implies that this reflection is Q, since it is the only point on ray $\overrightarrow{YQ}$ that lies on this circumscribed circle. In a similar way it can be established that R is the reflection of H over $\overline{AB}$.

Now note that

$$\begin{aligned} \angle QPR &= \angle YPX && (Y, X \text{ on rays } \overrightarrow{PQ}, \overrightarrow{PR}) \\ &= 180^\circ - \angle PXY - \angle XYP && (\text{Sum of angles in } \triangle PXY) \\ &= 180^\circ - 2\angle XAY && (\text{Tangent-chord theorem}) \\ &= 180^\circ - 2\angle BAC && (B, C \text{ on rays } \overrightarrow{AX}, \overrightarrow{AY}) \end{aligned}$$

implies that $\angle QPR = 180^\circ - 2\angle BAC$. From here, showing that $\angle QCR = 180^\circ - 2\angle BAC$ would be enough to prove that A, B, P and C lie on a circle because of the inverse of the inscribed angle theorem. This is just straightforward angle chasing and the solution is thus complete. □

Exercise 5.2. *Complete the angle chasing in the solution above by showing that* $\angle QCR = 180^\circ - 2\angle BAC$.

6 Conclusion

The results and examples discussed above have hopefully been both insightful and entertaining. The reader is highly encouraged to investigate the concepts of the *Euler line* and the *Nine point circle*, both of which are related to the orthocentre and the orthic triangle, but too extensive too do them justice in this short text. To conclude, a few practice problems are given for the reader to apply the ideas presented and build new problem solving strategies.

7 Exercises for the reader

1. Show that if H is the orthocentre of triangle ABC, then A is the orthocentre of triangle HBC. This is also called the orthocentric system, where four points on a plane always can form a triangle and its corresponding orthocentre.

2. Find four similar triangles with vertices among the points A, B, C, D, E, F, where $\triangle DEF$ is the orthic triangle of $\triangle ABC$.

3. Given a triangle ABC and its orthocentre H, prove that the reflections of H over the midpoints of segments AB, BC and CA all lie on the circumscribed circle of triangle ABC.

4. In triangle ABC with orthocentre H and midpoints K, L and M of sides AB, BC and CA respectively, the point P is chosen on the circumscribed

circle of $\triangle ABC$ such that $\angle HAP = 90°$. Show that the midpoint of segment HP lies on the circumscribed circle of $\triangle KLM$.

Extra challenging problem:

Prove that the orthic triangle has the smallest perimeter among the triangles that can be inscribed in an acute triangle. This problem is also called Fagnano's problem and is very difficult. The reader is encouraged to first try to prove the problem geometrically and then look up some famous solutions to the problem and attempt to complete those solutions. Some interesting geometric solutions have been carried out by Hermann Schwarz and Lipót Fejér.